CHAPTER 9

Projectile on Inclined Plane, Motion of Two Interacting Bodies and Bifilar Pendulum

9.1. RANGE ON AN INCLINED PLANE

A particle is projected with a velocity u at an angle α to the horizontal from a point O on an inclined plane, inclined at an angle β to the horizontal. The direction of projection lies in the vertical plane through OA the line of greatest slope, of the plane. Let the particle strike the inclined plane at A. Then OA (= R) is the range on the inclined plane (Fig. 9.1).

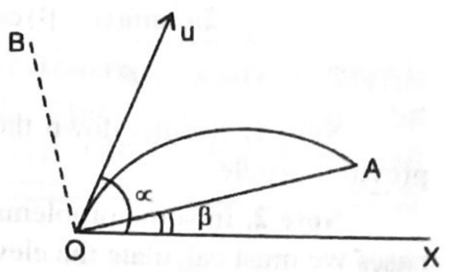

Fig. 9.1

Let OX and OA be respectively the horizontal and inclined plane through the point of projection O. OB is a line perpendicular to OA.

Component of initial velocity u along $OA = u\cos(\alpha - \beta)$

Component of initial velocity u along $OB = u\sin(\alpha - \beta)$

The projectile moves with a vertical retardation g.

Acceleration along $OA = -g\sin\beta$

Acceleration along $OB = -g\cos\beta$

Now, let T be the time taken by the particle to go from O to A. When the particle reaches A after time T, the distance moved perpendicular to the plane is zero.

Hence,
$$0 = u\sin(\alpha - \beta).T - \frac{1}{2}g\cos\beta.T^2 \qquad \left(\because\ s = ut + \frac{1}{2}at^2\right)$$

$$T = \frac{2u\sin(\alpha - \beta)}{g\cos\beta} \qquad \ldots(1)$$

When the particle strikes A after time T, the distance OA (= R) moved is the range on the inclined plane.

$$\therefore \quad R = u\cos(\alpha - \beta).T - \frac{1}{2}g\sin\beta.T^2$$

$$= u\cos(\alpha - \beta)\frac{2u\sin(\alpha - \beta)}{g\cos\beta} - \frac{1}{2}g\sin\beta\frac{4u^2\sin^2(\alpha - \beta)}{g^2\cos^2\beta}$$

$$= \frac{2u^2\sin(\alpha - \beta)}{g\cos^2\beta}[\cos(\alpha - \beta)\cos\beta - \sin(\alpha - \beta)\sin\beta]$$

$$= \frac{2u^2\sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}$$

9.2. RANGE AND TIME OF FLIGHT DOWN AN INCLINED PLANE

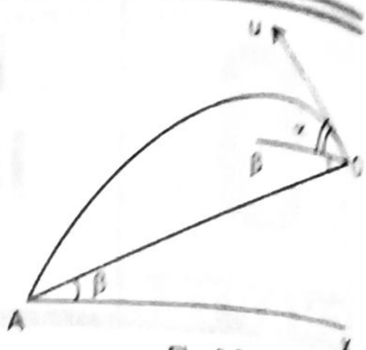
Fig. 9.2

The particle is projected down the plane from O at an elevation α [Fig. 9.2]. Initial velocities along and perpendicular to OA are $u\cos(\alpha+\beta)$ and $u\sin(\alpha+\beta)$. Acceleration along and perpendicular to OA are $g\sin\beta$ and $-g\cos\beta$. When the particle reaches A after time T_1, the distance moved perpendicular to the inclined plane is zero. Therefore,

$$0 = u\sin(\alpha+\beta).T_1 - \frac{1}{2}g\cos\beta.T_1^2 \text{ or } T_1 = \frac{2u\sin(\alpha+\beta)}{g\cos\beta}$$

$$\text{Range} = OA = R_1 = u\cos(\alpha+\beta).T_1 + \frac{1}{2}g\sin\beta.T_1^2$$

$$= \frac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$$

Note 1. Results down the plane can be obtained by putting $-\beta$ for β in the results of the previous article.

Note 2. In some problems, the elevation relative to the inclined plane may be given. In such cases we must calculate the elevation relative to the horizontal.

Example. *A particle is projected with a velocity of 32 ms^{-1} at an angle of 60° to the horizontal. Find the range on a plane inclined at 30° to the horizontal when projected (i) up the plane and (ii) down the plane.*

(*i*) When the particle is projected up the plane, the range on inclined plane is given by

$$R = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

Here, $u = 32$ ms^{-1} ; $\alpha = 60°$; $\beta = 30°$; $g = 9.8$ ms^{-2}

$$\therefore \quad R = \frac{2\times 32\times 32\times\sin 30°\times\cos 60°}{9.8\cos^2 30°} = 69.66 \text{ m.}$$

(*ii*) When the particle is projected down the plane at an angle α with the horizontal, the range down the plane is given by

$$R_1 = \frac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$$

$$= \frac{2\times 32\times 32\times\sin 90°\times\cos 60°}{9.8\cos^2 30°} = 139.3 \text{ m}$$

Maximum Range : *To find the direction of projection for the maximum range on the inclined plane.*

$$R = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

$$= \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta]$$

The value of R depends on α, for given values of u and β. Hence R is a maximum when $\sin(2\alpha-\beta) = 1$; *i.e.*, when $2\alpha - 2\beta = 90°$ or $\alpha = (45 + \beta/2)°$.

$\therefore$ The maximum range on the inclined plane

$$R_m = \frac{u^2}{g\cos^2\beta}(1-\sin\beta)$$

$$\therefore \quad R_m = \frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)} = \frac{u^2(1-\sin\beta)}{g(1+\sin\beta)(1-\sin\beta)}$$

$$= \frac{u^2}{g(1+\sin\beta)} \quad ...(3)$$

Note : $\alpha = 45° + \beta/2$. Then $\alpha - \beta = 45 - \beta/2$ and $90° - \alpha = 45° - \beta/2$.

This shows that the direction giving the maximum range bisects the angle between the vertical and the inclined plane.

9.3. TO SHOW THAT FOR A GIVEN VELOCITY OF PROJECTION AND A GIVEN RANGE ON THE INCLINED PLANE THERE ARE TWO DIRECTIONS OF PROJECTION WHICH ARE EQUALLY INCLINED TO THE DIRECTION FOR THE MAXIMUM RANGE

Now, $$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta) - \sin\beta]$$

The range R and the values of u and β are given.

Hence $\sin(2\alpha - \beta)$ is constant. There are two values of $(2\alpha - \beta)$, each less than 180° which satisfy the above equation. Let the corresponding values of α be α_1 and α_2. Then

$$2\alpha_1 - \beta = 180 - (2\alpha_2 - \beta) \text{ or } \alpha_1 - \beta/2 = 90 - (\alpha_2 - \beta/2)$$

$$\alpha_1 - (45 + \beta/2) = (45 + \beta/2) - \alpha_2.$$

$(45 + \beta/2)$ is the angle of projection giving the maximum range. Therefore, it follows that the direction giving maximum range bisects the angle between the two angles of projection that can give a particular range (Fig. 9.3).

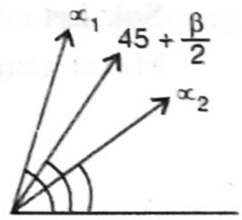

Fig. 9.3

Example 1. *Prove that for a given velocity of projection the ratio between the maximum ranges up and down an inclined plane inclined at an angle β to the horizon is* $\frac{1-\sin\beta}{1+\sin\beta}$.

We have already proved that,

Maximum range up the inclined plane $= R_m = \dfrac{u^2}{g(1+\sin\beta)}$

Range down an inclined plane $= R_1 = \dfrac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$

$$= \frac{u^2}{g\cos^2\beta}[\sin(2\alpha+\beta) + \sin\beta]$$

This range is maximum when $\sin(2\alpha + \beta) = 1$.

Maximum range down the inclined plane $= R_{m1} = \dfrac{u^2(1+\sin\beta)}{g\cos^2\beta} = \dfrac{u^2}{g(1-\sin\beta)}$

Hence $$\frac{R_m}{R_{m1}} = \frac{u^2}{g\,(1+\sin\beta)} \times \frac{g\,(1-\sin\beta)}{u^2} = \frac{1-\sin\beta}{1+\sin\beta}$$

Example 2. *A particle is projected at an angle α with the horizontal from the foot of the plane, whose inclination to the horizontal is β. Show that it will strike the plane at right angles if* $\cot\beta = 2\tan(\alpha-\beta)$.

Sol. Let u be the velocity of projection. The components of u parallel and perpendicular to the plane are $u\cos(\alpha-\beta)$ and $u\sin(\alpha-\beta)$ respectively. Accelerations in these directions are $g\sin\beta$ and $g\cos\beta$ respectively (Fig. 9.4).

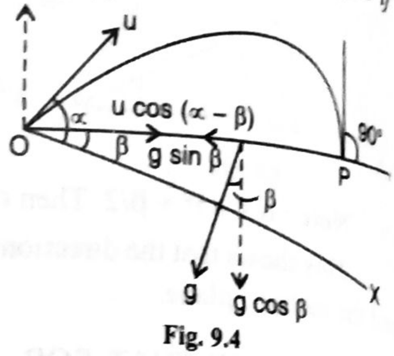

Fig. 9.4

Time of flight on the inclined plane

$$= T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \quad ...(1)$$

Since the particle hits the plane at right angles to the plane, the velocity of the particle parallel to the plane at that instant is zero.

$$0 = u\cos(\alpha-\beta) - g\sin\beta.\, T \text{ or } T = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad ...(2)$$

Equating (1) and (2), $$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

or $$\cot\beta = 2\tan(\alpha-\beta).$$

Example 3. *The angular elevation of an enemy's position on a hill h metres high is β. Show that in order to shell it, the initial velocity of the projectile must not be less than* $\sqrt{gh(1+\operatorname{cosec}\beta)}$.

Sol. Let u be the velocity of projection.

Maxm. range up a plane inclined at an angle α with the horizontal

$$= R_m = \frac{u^2}{g\,(1+\sin\beta)}$$

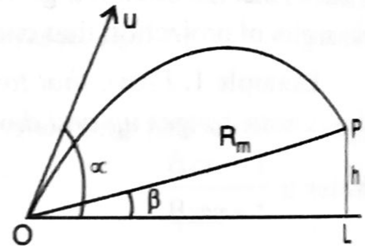

Fig. 9.5

If the shell is to hit enemy's position $R_m = \frac{h}{\sin\beta}$ (Fig. 9.5)

$$\therefore \quad \frac{u^2}{g\,(1+\sin\beta)} = \frac{h}{\sin\beta}$$

$$\therefore \quad u^2 \geq \frac{gh(1+\sin\beta)}{\sin\beta}$$

or $$u \geq \sqrt{gh(1+\operatorname{cosec}\beta)}$$

Example 4. *The greatest range on the horizontal plane of a projectile starting with a certain velocity is 150 m. What is the greatest range when projected with the same velocity up an inclined plane inclined 30° to the horizon ?*

Sol. Let u be the velocity of projection.

Max. range on the horizontal plane $= \frac{u^2}{g} = 150\ m.$

Max. range on the inclined plane

$$= \frac{u^2}{g\,(1+\sin\beta)} = \frac{150}{1+\sin 30^\circ} = 100 \text{ m}$$

8 Impact of Elastic Bodies

CHAPTER

8.1. IMPULSE OF A FORCE

The impulse I of a constant force F acting for a time t is defined as $F \times t$.

$$I = F \times t.$$

By Newton's second law, $F = ma$.

If u and v are the initial and final velocities of the particle,

$$a = (v - u)/t$$

$$\therefore \quad I = Ft = mat = m\left(\frac{v-u}{t}\right)t = m(v-u)$$

Thus *the impulse of a force is equal to the change in momentum produced.*

Impulsive Force : Definition. *An impulsive force is an infinitely great force acting for a very short interval of time, such that their product is finite.*

The force and the time cannot be measured because one is too great and the other is too small. Nevertheless, their product, which is definite, is capable of measurement. This we have seen, is the impulse of the impulsive force and is equal to the change in momentum produced. Hence an impulsive force is always measured by the change in momentum produced. In practice, the conditions of an impulsive force are never realized. Some approximate examples of impulsive force are : (1) the blow of a hammer on a pile and (2) the force exerted by the bat on a cricket ball.

Example 1. *A pile driver of mass 3000 kg falls through a height of 5m on to a pile of mass 1000 kg. If the pile is driven 0.24 m into the ground, find the resistance of the ground (supposed uniform).*

Sol. Velocity with which the pile driver impinges on the pile

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 5} = 9.9 \text{ ms}^{-1}$$

Momentum of the pile driver = $mv = 3000 \times 9.9$ kg ms^{-1}.

After the impact the total mass in motion is 4000 kg.

Let V be the common velocity of the driver and pile immediately after the impact.

By the principle of conservation of momentum,

$$4000 \times V = 3000 \times 9.9 \text{ or } V = 7.4 \text{ ms}^{-1}.$$

Now the velocity is destroyed in 0.24 m. If a is the retardation, we get

$$0 = (7.4)^2 - 2a\,(0.24) \text{ or } a = 114.1 \text{ ms}^{-2}$$

The retarding force is $4000 \times 114.1 = 4.56 \times 10^5$ N

The resistance of the ground R must equal the retarding force plus the weight of the driver and pile.

$$\therefore \quad R = 4.56 \times 10^5 + 4000 \times 9.8 = 4.95 \times 10^5 \text{ N}$$

8.2. WHAT IS A COLLISION ?

We learn much about atomic, nuclear and elementary particles, experimentally by observing collisions between them. The study of collisions of molecules in gases has developed into kinetic theory of gases. The study of collisions is based on the principles of conservation of momentum and conservation of energy.

In a collision, a relatively large force acts on each colliding particle for a relatively short time. The force is called an *impulsive force*. When a bat hits a ball, the bat exerts a large force on the ball. Both the ball and the bat are deformed during the collision.

The force of interaction may be due to different causes in different cases.

Thus in the collision between two billiard balls, the force of interaction is due to elasticity. It comes into existence only when the two billiard balls come into physical contact. In the case of scattering of α particles by the nuclei, it is the electrostatic force that causes the interaction.

An alpha particle projected towards the nucleus of an atom will be repelled by the electrostatic forces due to the nucleus. In this case, the particles will not touch each other. Even then this process is called a collision because a relatively strong force acts between the particles and this force has a marked effect on the motion of the α particle.

Elastic and Inelastic collisions : There are two types of collision :

(*i*) elastic and (*ii*) inelastic.

(*i*) Elastic collisions are those in which the total kinetic energy before and after the collision remains unchanged. Collisions between atomic, nuclear and fundamental particles are the true elastic collisions. Collisions between ivory or glass balls can be treated as approximately elastic collisions. In such a collision between two particles, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

and $$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

where m_1 and m_2 are the respective masses of the two particles and u_1, u_2 and v_1, v_2 their velocities *before* and *after* the collision.

(*ii*) If the K. E. is not conserved, the collision is said to be inelastic. When two bodies stick together after collision, the collision is said to be completely inelastic. For example, the collision between a bullet and its target is completely inelastic when the bullet remains embedded in the target.

Completely Inelastic Collision : Suppose a body of mass m_1 moving with a velocity u_1 collides with a body of mass m_2 moving with velocity u_2 in the same direction. The two bodies stick together after collision and they move with a final common velocity V in the same direction as the original. It is not necessary to restrict the discussion to one dimensional motion. Using only the conservation of momentum principle,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\,V.$$

From this the value of V can be determined if u_1 and u_2 are known.

FUNDAMENTAL PRINCIPLES OF IMPACT

1. **Newton's law of impact-coefficient of restitution.** When two bodies impinge directly, their relative velocity after impact is in a constant ratio to their relative velocity before impact and is in the opposite direction. This constant ratio depends only on the material of the bodies and not on their masses or velocities. It is called the coefficient of restitution and is denoted by the letter e. If u_1, u_2 be the velocities of two bodies before the impact and v_1, v_2 the velocities after impact,

$$\frac{v_1 - v_2}{u_1 - u_2} = -e \text{ or } v_1 - v_2 = -e\,(u_1 - u_2)$$

Here, $(u_1 - u_2)$ and $(v_1 - v_2)$ are their relative velocities, before and after the impact. e lies between 0 and 1. If $e = 0$, the bodies are called perfectly plastic bodies. If $e = 1$, the bodies are called perfectly elastic bodies. For two glass balls, $e = 0.94$; For two lead balls, $e = 0.2$.

Definition of coefficient of restitution. *The ratio, with a negative sign, of the relative velocity of two bodies after impact to their relative velocity before impact is called the coefficient of restitution.*

2. Motion of two smooth bodies perpendicular to the line of impact. When two smooth bodies impinge, there is no tangential action between them. Hence there is no change of momentum along the common tangent. Hence, there is no change of velocity for either body along the tangent. In other words, there is no change in the velocity of a body in a direction perpendicular to the common normal due to impact.

3. Principle of conservation of momentum. The total momentum of two bodies after impact along the *common normal* should be equal to the total momentum before the impact along the same direction.

The above three principles are sufficient to determine the change in motion of two impinging smooth bodies.

Definitions. (*i*) Two bodies are said to impinge *directly* when the direction of motion of each is along the common normal at the point where they touch.

(*ii*) Two bodies are said to impinge *obliquely* if the direction of motion of either or both is not along the common normal at the point of contact.

(*iii*) The common normal at the point of contact is called the line of impact. Thus, in the case of two spheres the line of impact is the line joining their centres.

8.3. OBLIQUE IMPACT OF A SMOOTH SPHERE ON A FIXED SMOOTH PLANE

Let XY be the fixed plane. Let the sphere strike the fixed plane at point P. Then if C is the centre of the sphere, CP is the common normal at the point of contact of the plane and the sphere. Let u and v be the velocities of the sphere before and after impact making angles α and θ respectively with the common normal CP [Fig. 8.1]. By Newton's experimental law, the relative velocity of the sphere along the common normal after impact is $-e$ times its relative velocity along the common normal before impact.

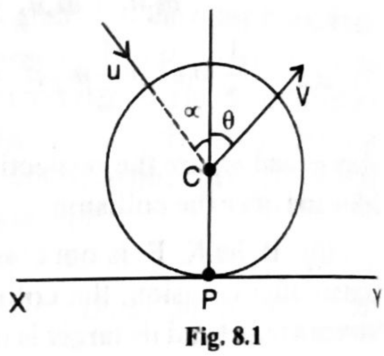

Fig. 8.1

$$v \cos\theta - 0 = -e(-u\cos\alpha - 0)$$

or $$v\cos\theta = eu\cos\alpha \quad ...(1)$$

Since both the sphere and the plane are smooth, there is no force in a direction parallel to the plane. Hence the velocity of the sphere resolved parallel to the plane is unaltered by the impact.

$\therefore$ $$v\sin\theta = u\sin\alpha \quad ...(2)$$

Dividing (1) by (2), $\cot\theta = e\cot\alpha$ ...(3)

Squaring and adding (1) and (2),

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha) \quad ...(4)$$

Equations (4) and (3) give the velocity and direction of the sphere after impact.

Cor. 1. The impulse of the plane on the sphere is measured by the change of momentum of the sphere measured along the normal.

$$I = m[v\cos\theta - (-u\cos\alpha)] = m[v\cos\theta + u\cos\alpha]$$

$$= m[eu\cos\alpha + u\cos\alpha]$$

$\therefore$ $$I = mu(1+e)\cos\alpha$$

Cor. 2. If $e = 1$, then $\theta = \alpha$ and $v = u$ *i.e.*, when a perfectly elastic sphere impinges on a fixed mooth plane its velocity is unaltered in magnitude by the impact and the angle of reflection is equal the angle of incidence.

Cor. 3. If $e = 0$, then $\cot\theta = 0$ or $\theta = 90°$ [from (3)] and $v = u \sin\alpha$ [from (4)]. Thus if the sphere nd the plane are both inelastic, the sphere moves along the plane with velocity $u \sin\alpha$.

Cor. 4. If $\alpha = 0$ then $\theta = 0$ and $v = eu$ from (3) and (4) *i.e.*, if a sphere impinges normally on a orizontal plane, it rebounds vertically with velocity eu.

Cor. 5. The change in K.E. of the sphere due to impact on the plane is given by

$$\frac{1}{2}m(v^2 - u^2) = \frac{1}{2}m(v+u)(v-u) = \frac{1}{2}I(v+u)$$

Here, $m(v-u) = I$ = Impulse of the force of the sphere on the plane.

Example 1. *A steel ball is let fall through a height of 0.64 m on a plate of steel. The height hrough which it rebounds is 0.36 m. Calculate the coefficient of restitution.*

Sol. Let u be the velocity of the ball when it strikes the plane. Then $u^2 = 2gh$. The ball rebounds vertically with velocity eu after the first impact. The height to which the particle rebounds after the first impact $= \dfrac{(eu)^2}{2g} = h_1$; *i.e.*, $h_1 = \dfrac{e^2u^2}{2g}$

But $\qquad u^2 = 2gh \;\therefore\; h_1 = e^2 h$ or $e^2 = \dfrac{h_1}{h}$ or $e = \sqrt{\dfrac{h_1}{h}}$

Here, $\qquad h_1 = 0.36\ m;\; h = 0.64\ m;\; \therefore\; e = \sqrt{\dfrac{0.36}{0.64}} = 0.75$

8.4. DIRECT IMPACT OF TWO SMOOTH SPHERES

A smooth sphere of mass m_1 moving with a velocity u_1 impinges on another smooth sphere of mass m_2 moving in the same direction with velocity u_2. If e is the coefficient of restitution between them, find the velocities of the spheres after impact.

Since the spheres are smooth, there is no impulsive force on either along the common tangent. Hence in this direction their velocities after impact are the same as their original velocities *i.e.*, zeroes.

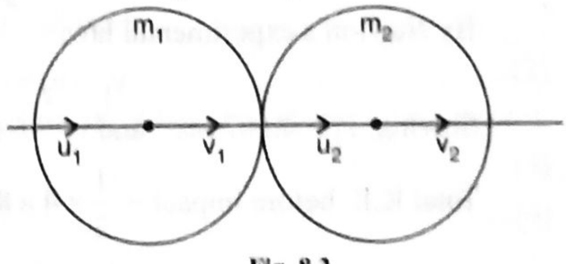

Fig. 8.2

Let v_1 and v_2 be the velocities of the two spheres along the common normal after impact [Fig. 8.2].

By the principle of conservation of momentum,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \qquad \text{....(1)}$$

By Newton's experimental low,

$$v_1 - v_2 = -e(u_1 - u_2) \qquad \text{...(2)}$$

Multiplying (2) by m_2 and adding to (1),

$$v_1(m_1 + m_2) = m_2u_2(1+e) + u_1(m_1 - em_2)$$

$$\therefore \quad v_1 = \frac{m_2u_2(1+e) + u_1(m_1 - em_2)}{(m_1 + m_2)} \qquad \text{...(3)}$$

Multiplying (2) by m_1 and subtracting from (1),

$$v_2(m_1 + m_2) = m_1u_1(1+e) + u_2(m_2 - em_1)$$

$$v_2 = \frac{m_1 u_1 (1 + e) + u_2 (m_2 - e m_1)}{(m_1 + m_2)} \quad ...(4)$$

Equations (3) and (4) give the velocities of the two spheres after impact.

Cor. 1. The impulse of the blow on the sphere of mass m_1 = change of momentum produced in it $= m_1 (v_1 - u_1) = \dfrac{m_1 m_2 (1 + e)(u_2 - u_1)}{m_1 + m_2}$.

This is equal and opposite to the impulse of the blow on the sphere of mass m_2.

Cor. 2. If $e = 1$ and $m_1 = m_2$ then, $v_1 = u_2$ and $v_2 = u_1$. Thus, if two equal perfectly, elastic spheres impinge directly, they interchange their velocities.

Example 1. *A ball of mass 8 kg, moving with a velocity of 10 ms^{-1} impinges directly on another mass 24 kg, moving at 2 ms^{-1} in the opposite direction. If e = 0.5, find the velocities of the balls after impact.*

Sol. Let v_1 and v_2 be the velocities of the balls after impact (Fig. 8.3).

By the principle of conservation of momentum,

$$8v_1 + 24v_2 = 8 \times 10 - 24 \times 2$$

By Newton's experimental law,

$$v_1 - v_2 = -0.5\,[10 - (-2)]$$

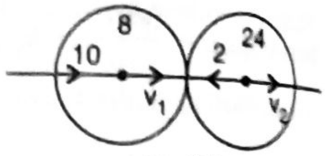

Fig. 8.3

Solving $v_1 = -3.5$ ms^{-1} and $v_2 = 2.5$ ms^{-1}

Example 2. *A smooth sphere of mass 4 kg moving with a velocity of 8 ms^{-1} impinges directly on a smooth sphere of mass 5kg moving in the same direction with a velocity of 4 ms^{-1}. Find the velocities of the spheres after impact. Calculate also the loss of K. E. due to the impact and the impulse of the blow on the sphere of smaller mass, (e = 0.5).*

Sol. Let v_1 and v_2 be the velocities of the smaller and bigger spheres after impact.

By the principle of conservation of momentum,

$$4v_1 + 5v_2 = 4 \times 8 + 5 \times 4 \quad ...(1)$$

By Newton's experimental law,

$$v_1 - v_2 = -0.5\,[8 - 4] \quad ...(2)$$

Solving, $v_1 = 4.667$ ms^{-1} and $v_2 = 6.667$ ms^{-1}

Total K.E. before impact $= \dfrac{1}{2} \times 4 \times 8^2 + \dfrac{1}{2} \times 5 \times 4^2 = 168$ J

Total K.E. after impact

$$= \frac{1}{2} \times 4 \times (4.667)^2 + \frac{1}{2} \times 5 \times (6.667)^2 = 154.7 \text{ J}$$

Loss of K.E. = 168 – 154.7 = 13.3 J.

Impulse of the blow on the sphere of mass 4 kg

$$= m(v_1 - u_1) = 4(4.667 - 8) = -13.33 \text{ J}.$$

Example 3. *A ball of mass m_1 moving with velocity u_1 strikes another ball of mass m_2 which is stationary. If the collision is elastic, calculate the fraction of the kinetic energy transferred to the second ball.*

Sol. From Eq. (4), $v_2 = \dfrac{2m_1 u_1}{(m_1 + m_2)}$ ($\because u_2 = 0$ and $e = 1$)

Initial K. E. of the first ball $= \frac{1}{2} m_1 u_1^2$.

K.E. transferred to the second ball $= \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 \left(\frac{2m_1 u_1}{m_1 + m_2} \right)^2$

Fraction of K.E. transferred to second ball

$$= \frac{\frac{1}{2} m_2 \left(\frac{2m_1 u_1}{m_1 + m_2} \right)^2}{\frac{1}{2} m_1 u_1^2} = \frac{4m_1 m_2}{(m_1 + m_2)^2}.$$

Example 4. *A ball impinges on another equal ball moving with same speed in a direction perpendicular to its own, the line joining the centres of the balls at the instant of impact being perpendicular to the direction of motion of the second ball; if e be the coefficient of restitution, show that the direction of motion of the second ball is turned through* $\tan^{-1}\left(\frac{1+e}{2}\right)$.

Sol. Let C_1 and C_2 be the centres of the balls A and B. Let u be the velocity of the ball A before impact along the line of centres and u the velocity of B, perpendicular to the line of centres. Since the velocity of the ball A is u along the line of centres, after impact it will move only along the line of centres. Let it move with velocity v. Let v_1 be the velocity of the second ball B at an angle θ with the line of centres [Fig. 8.4].

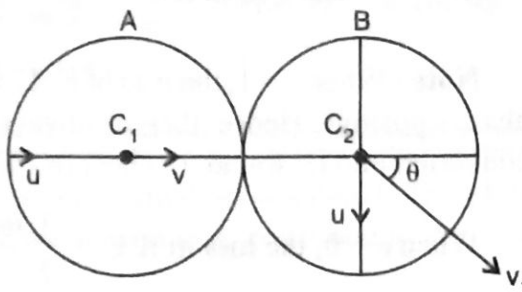

Fig. 8.4

By the principle of conservation of momentum,

$$mv_1 \cos\theta + mv = mu \quad \text{where } m \text{ is the mass of each ball}$$

or $$v_1 \cos\theta + v = u \qquad ...(1)$$

By Newton's experimental law, $v_1 \cos\theta - v = -e(0 - u) = eu$

or $$v_1 \cos\theta - v = eu \qquad ...(2)$$

The velocity of the second ball B perpendicular to the line of centres is unaltered by the impact.

Hence, $$v_1 \sin\theta = u \qquad ...(3)$$

Adding (1) and (2), $2v_1 \cos\theta = u(1 + e)$ ...(4)

Dividing (4) by (3), $2 \cot\theta = (1 + e)$ or $\cot\theta = \frac{(1+e)}{2}$.

The angle through which the direction of motion of the second ball is turned through is $90° - \theta$, *i.e.*, $\tan^{-1} \frac{1+e}{2}$.

8.5. LOSS OF K.E. DUE TO DIRECT IMPACT OF TWO SMOOTH SPHERES

Let m_1, m_2 be the masses, u_1 and u_2, v_1 and v_2 their velocities before and after impact and e the coefficient of restitution. Then, by the principle of conservation of linear momentum,

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \qquad ...(1)$$

By Newton's experimental law,

$$v_1 - v_2 = -e(u_1 - u_2) \qquad ...(2)$$

Square both equations, multiply the square of the second by m_1m_2 and add the results. Then,

$$\left.\begin{matrix}(m_1^2 + m_1m_2)\, v_1^2 + \\ (m_2^2 + m_1\, m_2)\, v_2^2\end{matrix}\right\} = (m_1u_1 + m_2u_2)^2 + e^2 m_1m_2(u_1 - u_2)^2$$

$$\therefore \quad m_1(m_1 + m_2)v_1^2 + m_2(m_1 + m_2)v_2^2 = (m_1u_1 + m_2u_2)^2 + m_1m_2(u_1 - u_2)^2 + e^2m_1m_2(u_1 - u_2)^2 - m_1m_2(u_1 - u_2)^2$$

$$\therefore \quad (m_1 + m_2)(m_1v_1^2 + m_2v_2^2) = (m_1 + m_2)(m_1u_1^2 + m_2u_2^2) - m_1m_2(u_1 - u_2)^2(1 - e^2)$$

$$\therefore \quad \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(u_1 - u_2)^2(1 - e^2)$$

Now, $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 =$ K.E. after impact.

$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 =$ K.E. before impact.

$\therefore$ The loss in K.E. $= \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(u_1 - u_2)^2(1 - e^2)$

Note : When $e = 1$, the loss of K. E. is zero. In general $e < 1$ so that $(1 - e^2)$ is positive. $(u_1 - u_2)^2$ is always positive. Hence, there is always a loss of K.E. due to impact. The K.E. lost during impact is converted into (*i*) sound, (*ii*) heat or (*iii*) vibration or rotation of the colliding bodies.

When $e = 0$, the loss in K.E. $= \frac{1}{2}\frac{m_1m_2(u_1 - u_2)^2}{(m_1 - m_2)}$

i.e., there is maximum loss of K.E. on impact of plastic bodies.

8.6. OBLIQUE IMPACT OF TWO SMOOTH SPHERES

A smooth sphere of mass m_1 moving with velocity u_1 impinges obliquely on a smooth sphere of mass m_2 moving with velocity u_2. If the directions of motion before impact make angles α and β with the common normal, find the velocities and direction of the spheres after impact.

Let *AB* be the common normal (Fig.8.5). Let v_1 and v_2 be the velocities of the two spheres after impact making angles θ and ϕ with the common normal *AB*. *Before impact* velocities along the common normal *AB* are u_1 cos α and u_2 cos β and velocities perpendicular to *AB* are u_1 sin α and u_2 sin β. After impact velocities along *AB* are v_1 cos θ and v_2 cos ϕ and perpendicular to *AB* are v_1 sin θ and v_2 sin ϕ.

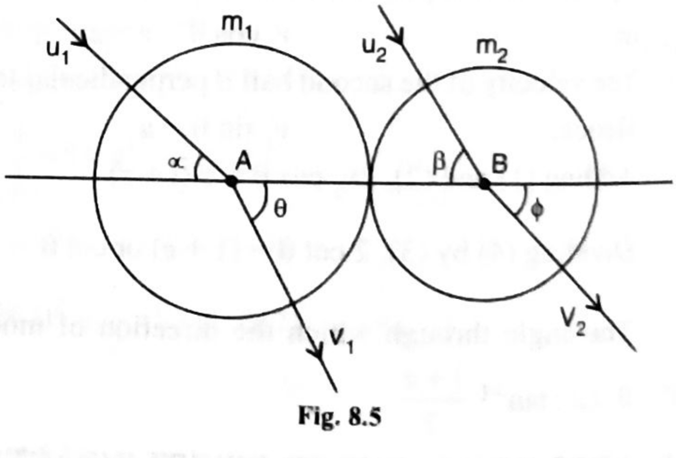

Fig. 8.5

By the principle of conservation of momentum, the total momentum of the two spheres along the common normal is unaltered by the impact.

$$\therefore \quad m_1v_1\cos\theta + m_2v_2\cos\phi = m_1u_1\cos\alpha + m_2u_2\cos\beta \qquad ...(1)$$

By Newton's experimental law (on relative velocities along the common normal),

$$v_1\cos\theta - v_2\cos\phi = -e(u_1\cos\alpha - u_2\cos\beta) \qquad ...(2)$$

Since there is no force perpendicular to the common normal *AB*, the velocities of the spheres perpendicular to the common normal *AB* remain unaltered due to impact. Hence

$$v_1 \sin\theta = u_1 \sin\alpha \quad ...(3)$$

and

$$v_2 \sin\phi = u_2 \sin\beta \quad ...(4)$$

Multiplying (2) by m_2 and adding to (1),

$$v_1 \cos\theta = \frac{(m_1 - em_2)\, u_1 \cos\alpha + m_2\,(1+e)\, u_2 \cos\beta}{(m_1 + m_2)} \quad ...(5)$$

Multiplying (2) by m_1 and subtracting from (1),

$$v_2 \cos\phi = \frac{(m_2 - em_1)\, u_2 \cos\beta + m_1\,(1+e)\, u_1 \cos\alpha}{(m_1 + m_2)} \quad ...(6)$$

Squaring (3) and (5) and adding we get v_1^2 and hence we can find v_1. Dividing (3) by (5) we get $\tan\theta$. Similarly, from (4) and (6) we can get v_2 and $\tan\phi$. Therefore, v_1, v_2, θ and ϕ are determined uniquely.

Cor. 1. The impulse of the blow on the sphere of mass m_1 = its change of momentum measured along the common normal

$$= m_1 v_1 \cos\theta - m_1 u_1 \cos\alpha$$

$$= m_1\,(v_1 \cos\theta - u_1 \cos\alpha) = \frac{m_1\, m_2\,(1+e)}{(m_1 + m_2)}\,(u_2 \cos\beta - u_1 \cos\alpha)$$

This is equal and opposite to the impulse on the sphere of mass m_2.

8.7. LOSS OF K.E. DUE TO OBLIQUE IMPACT

The velocities of the spheres perpendicular to the common normal are unaltered. Therefore, the loss of K.E. is the same as in the case of direct impact if we substitute $u_1 \cos\alpha$ and $u_2 \cos\beta$ for u_1 and u_2 respectively.

$$\therefore \quad \text{The loss in K.E.} = \frac{m_1 m_2\,(1-e^2)}{2\,(m_1 + m_2)}\,(u_1 \cos\alpha - u_2 \cos\beta)^2$$

Example 1. *A ball of mass m impinges obliquely on a ball of mass M at rest. If m = eM, prove that the directions of motion of the balls are at right angles after impact.*

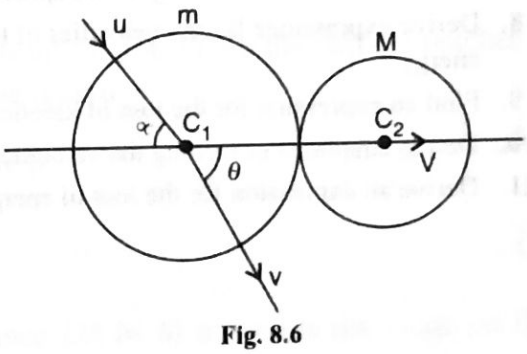

Fig. 8.6

Sol. Before impact, let u be the velocity of the ball of mass m and α its direction of motion with the common normal (Fig. 8.6). Before impact, the second ball of mass M is at rest. After impact, the second ball of mass M will move along the common normal, because the force on it during the period of impact is only along the common normal. Let the velocity of the second ball of mass M be V. After impact, let v be the velocity of the ball of mass m making an angle θ with the common normal.

By the principle of conservation of momentum,

$$mv \cos\theta + MV = mu \cos\alpha + 0$$

i.e.,

$$eMv \cos\theta + MV = eMu \cos\alpha$$

or,

$$ev \cos\theta + V = eu \cos\alpha \quad ...(1)$$

By Newton's experimental law,

$$v \cos\theta - V = -e\,(u \cos\alpha - 0) \text{ or } v \cos\theta - V = -eu \cos\alpha \quad ...(2)$$

Adding (1) and (2), $(1+e)\, v \cos\theta = 0$ or $\cos\theta = 0$ or $\theta = 90°$.

Hence, the directions of motion of the balls are at right angles after impact.

Example 2. *If an oblique collision occurs between two equal smooth perfectly elastic spheres one of which is initially at rest, show that their paths after impact are at right angles to one another.*

Sol. Let u_1 be the velocity and α_1 inclination to the line of impact of A before impact (Fig. 8.7). After impact, let these be v_1 and β_1. B is initially at rest. So after impact it moves along the line of impact with velocity v_2. By the principle of conservation of momentum,

$$mv_1 \cos \beta_1 + mv_2 = mu_1 \cos \alpha_1$$

or $$v_1 \cos \beta_1 + v_2 = u_1 \cos \alpha_1 \quad ...(1)$$

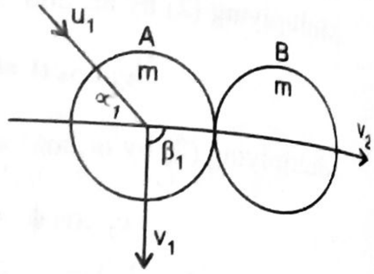

Fig. 8.7

By Newton's experimental law,

$$v_1 \cos \beta_1 - v_2 = -e\,(u_1 \cos \alpha_1 - 0) \quad ...(2)$$

Adding (1) and (2),

$$2v_1 \cos \beta_1 = (1 - e)\,(u_1 \cos \alpha_1) = 0 \; (\because e = 1)$$

v_1 can not be zero.

$\therefore$ $$\cos \beta_1 = 0 \text{ or } \beta_1 = 90°.$$

EXERCISE VIII

Section A

1. Mention the three principles that hold good when an impact takes place between two smooth spheres. Define "coefficient of restitution".
2. Define impulse of a force. How is it measured? What is an impulsive force?
3. Define : 'Line of impact". Distinguish between direct and oblique impacts.
4. Explain "Elastic collision"; "Inelastic collision".
5. Explain the momentum and impulse.
6. State and explain the difference between elastic and inelastic collision.

Section B

7. A smooth sphere has an oblique impact on a fixed smooth plane. Derive expressions for the magnitude and direction of the velocity of the sphere after impact.
8. Derive expressions for the velocities of two smooth spheres after direct impact. Find the loss of kinetic energy.
9. Find an expression for the loss of kinetic energy due to direct collision between two spheres.
10. Derive equations governing the velocities and directions of two smooth spheres after oblique impact.
11. Derive an expression for the loss of energy during the oblique impact of two solid spheres.