19UMAA01

Algebra and Calculus

B.Sc. Mathematics

I - Semester

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ALLIED MATHEMATICS - I

unit - I

Theory of equation

Imaginary roots - Irrational root - Formation of equation - Solution of equation - Diminishing the roots of an equation and Solutions - Removal of the Second term of an equation and Solutions - Descarte's rule of sign - problems only

unit - II

Matrices

Definition of charateristic equation of a matrix characteristic roots of a matrix - Eigen values and the corresponding eigen vectors of matrix - cayley hamilton theorem (statement only) - Verification of cayley Hamilton theorem problems only.

Unit - III

Radius of curvature

Formula of radius of curvature in cartesian coordinates paramatric coordinates and polar coordinates (No proof for formulae) - problems only.

Unit - IV

Partial Differential equation

Formation of partical differentical equation by eliminating the arbitr ary constants arbitrary function laquages linear partial differential equation - problemonly unit - V

Integration

Definite integral - Simple properties. of definite integrals - Bernoulli's formula Integration by parts - Simple problems - reduction formula $\int_0^{\pi/2} \sin^n x \, dx$ $\int_0^{\pi/2} \cos^n x \, dx$

$\int_0^{\infty} e^{-x} x^n \, dx$, $\int x^n e^{ax} \, dx$ - Simple problems

Polynomial equation

Imaginary and Irrational Roots

theorem:

In a polynomial equation $f(x) = 0$ with its real coefficients Imaginary roots occur in pairs.

theorem 2. In a polynomial equation $f(x)=0$ with real equation coefficients irrational roots occur in Pair's

$$\begin{array}{r|l} & x^2+4x+5 \\ \hline x^2-4x+7 & x^4+0x^3-4x^2+8x+35 \\ & x^4-4x^3+7x^2 \\ & (-)\ (+)\ (-) \\ \hline & 4x^3-11x^2+8x \\ & 4x^3-16x^2+28x \\ & (-)\ (+)\ (-) \\ \hline & 5x^2+\cancel{26x}-20x+35 \\ & 5x^2-20x+35 \\ \hline & 0 \end{array}$$

$\therefore$ the other roots are given by

$x^2+4x+5=0$

$a=1$
$b=4$
$c=5$

$$x = \frac{-4 \pm \sqrt{4^2-4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm \sqrt{i^2 4}}{2}$$

$$= \frac{-4 \pm i2}{2}$$

$$= \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$$= -2 \pm i$$

$\therefore$ The root are $2 \pm i\sqrt{3}$, $-2 \pm i$

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(2) Solve the eqn $x^4-11x^2+2x+12=0$ given that $\sqrt{5}-1$ is a root

Sol: Given eqn is $x^4-11x^2+2x+12=0$ $\rightarrow$ (1)

Given the $\sqrt{5}-1$ is a root of eqn (1)

is $-1+\sqrt{5}$ is a root of eqn (1)

Irrational roots always occur in pairs

$\therefore -1-\sqrt{5}$ is always a roots of eqn

$\therefore$ The real factors corresponding to there root I

$$[x-(-1+\sqrt{5})]\,[x-(-1-\sqrt{5})]$$

$$=[x+1-\sqrt{5}]\,[x+1+\sqrt{5}]$$

$$=[(x+1)-\sqrt{5}]\,[(x+1)+\sqrt{5}]$$

$$=(x+1)^2-(\sqrt{5})^2$$

$$=x^2+2x+1-5$$

$$=x^2+2x-4$$

$\therefore$ Dividing $x^4-11x^2+2x+12$ by x^2+2x-4 we get

$$\begin{array}{r|l}
 & x^2-2x \\
\hline
x^2+2x-4 & x^4-11x^2+2x+12 \\
 & x^4+2x^3-4x^2 \\
 & (-)\quad(-)\quad + \\
\hline
 & -2x^3-7x^2+2x \\
 & -2x^3-4x^2+8x \\
 & (+)\quad(+)\quad - \\
\hline
 & -3x^2-6x+12 \\
 & -3x^2-6x+12 \\
\hline
 & 0
\end{array}$$

$\therefore$ The other roots are given by $x^2-2x-3=0$

$$(x+1)(x-3)=0$$

$$x=-1,3$$

$\therefore$ the root of the given eqn are $\sqrt{5}-1, -\sqrt{5}-1, -1, 3$

Q) From the equation one of whose roots as $\sqrt{3}+\sqrt{5}$

sol: Given that $\sqrt{3}+\sqrt{5}$ is a root

Irrational Roots always occur in pairs

$\therefore \sqrt{3}-\sqrt{5}, -\sqrt{3}+\sqrt{5}, -\sqrt{3}-\sqrt{5}$ are also roots of the of the required equation

$\therefore$ the required equation is

$[x-(\sqrt{3}+\sqrt{5})][x-(\sqrt{3}-\sqrt{5})][x-(-\sqrt{3}+\sqrt{5})][x-(-\sqrt{3}-\sqrt{5})]$

is $[x-\sqrt{3}-\sqrt{5}][x-\sqrt{3}+\sqrt{5}][x+\sqrt{3}-\sqrt{5}][x+\sqrt{3}+\sqrt{5}]=0$

$[(x-\sqrt{3})^2-(\sqrt{5})^2][(x+\sqrt{3})^2-(\sqrt{5})^2]$

$[x^2-2\sqrt{3}x-2)(x^2+2\sqrt{3}x-2)=0$

$[(x^2-2)-2\sqrt{3}x][(x^2-2)+2\sqrt{3}x]=0$

$(x^2-2)^2-(2\sqrt{3}x)^2=0$

$x^4-4x^2+4-12x^2=0$

$x^4-16x^2+4=0$

which is the required eqn

Solve $x^5-x^4+8x^2-9x-15=0$ given that $-\sqrt{3}$ and $1+2\sqrt{-1}$ are roots

sol:

given eqn is $x^5-x^4+8x^2-9x-15=0$

Given that

Irrational roots and imaginary roots also occur in pairs

Given that $-\sqrt{3}$ is a root of eqn ①

$\Rightarrow \sqrt{3}$ is also a root of eqn ①

Also given that $1+2\sqrt{-1}$ is a root is $1+2i$ is a root

$\Rightarrow 1-2i$ is a also a root of eqn ①

∴ The real factors corresponding to these roots is

$= [x-(-\sqrt{3})][x-\sqrt{3}][x-(1+2i)][x-(1-2i)]$

$= (x+\sqrt{3})(x-\sqrt{3})[(x-1)-2i][(x-1)+2i]$

$= [x^2-(\sqrt{3})^2][(x-1)^2-(2i)^2]$

$= (x^2-3)(x^2-2x+1+4)$

$= (x^2-3)(x^2-2x+5)$

$= x^4-2x^3+5x^2-3x^2+6x-15$

$= x^4-2x^3+2x^2+6x-15$

Dividing $x^5-x^4+8x^2+9x-15$ by $x^4-2x^3+2x^2+6x-15$

$$
\begin{array}{r|l}
 & x+1 \\
\hline
x^4-2x^3+2x^2+6x-15 & x^5-x^4+8x^2+9x-15 \\
 & x^5-2x^4+2x^3+6x^2+15x \\
 & -\quad +\quad -\quad -\quad - \\
\hline
 & \quad x^4-2x^3+2x^2+6x-15 \\
 & \quad x^4-2x^3+2x^2+6x-15 \\
\hline
 & \quad\quad 0
\end{array}
$$

∴ The other root is given by

$x+1=0$

$\Rightarrow x=-1$

From the equation given that $-\sqrt{3}+i\sqrt{2}$ is a root

given that $-\sqrt{3}+i\sqrt{2}$ is a root

Irrational roots always occur in pairs

∴ $-\sqrt{3}-i\sqrt{2}$, $\sqrt{3}+i\sqrt{2}$, $\sqrt{3}-i\sqrt{2}$

are also root of the required equation

∴ The required eqn is

$$[x-(-\sqrt{3}+i\sqrt{2})][x-(-\sqrt{3}-i\sqrt{2})][x-(\sqrt{3}+i\sqrt{2})]$$
$$[x-(\sqrt{3}-i\sqrt{2})]$$

$$[x+\sqrt{3}-i\sqrt{2}][x+\sqrt{3}+i\sqrt{2}][x-\sqrt{3}-i\sqrt{2}]$$
$$[x-\sqrt{3}+i\sqrt{2}]=0$$

$$[(x+\sqrt{3})^2-(i\sqrt{2})^2][(x-\sqrt{3})^2-(i\sqrt{2})^2]=0$$

$$(x^2+2\sqrt{3}x+3+2)(x^2-2\sqrt{3}x+3+2)=0$$

$$(x^2+2\sqrt{3}x+5)(x^2-2\sqrt{3}x+5)=0$$

$$[(x^2+5)+2\sqrt{3}x][(x^2+5)-2\sqrt{3}x]=0$$

$$(x^2+5)^2-(2\sqrt{3}x)^2=0$$
$$x^4+10x^2+25-12x^2=0$$
$$x^4-2x^2+25=0$$

which is the required eqn

<u>Transformation of equation</u>

I If α, β, γ are the rubs of the equation $ax^3+bx^3+cx+d=0$ then

$$S_1=\alpha+\beta+\gamma=-\frac{b}{a}$$

$$S_2=\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{d}$$

$$S_3=\alpha\beta\gamma=-\frac{d}{a}$$

II If $\alpha, \beta, \gamma; \delta$ are the roots of the equation

$px^4+qx^3+rx^2+sx+t=0$ then

$$S_1=\alpha+\beta+\gamma+\delta=-\frac{q}{p}$$

$$S_2=\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\beta\delta+\alpha\gamma=\frac{r}{p}$$

$$S_3=\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{s}{p}$$

$$S_4=\alpha\beta\gamma\delta=\frac{t}{p}$$

① If α, β, γ are the roots of the equation $ax^3+bx^2+cx+d=0$ then find the equation whose roots are $\alpha^2, \beta^2, \gamma^2$

Sol:

Given that α, β, γ are the roots of the equation

$$ax^3+bx^2+cx+d=0 \rightarrow ①$$

$$\alpha+\beta+\gamma=-\frac{b}{a}$$

$$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$$

$$\alpha\beta\gamma=-\frac{d}{a}$$

$\Rightarrow y=\alpha^2$ since α is a roots of equation ① can be written as

$$ax^2 x+bx^2+cx+d=0$$

$$a\cdot yx+by+cx+d=0$$

$$\Rightarrow by+d=-axy-cx$$

$$by+d=-x\cdot(ay+c)$$

Squaring on bothsides we get

$$(by+d)^2=\{-x(ay+c)\}^2$$

$$(by+d)^2=x^2\cdot(ay+c)^2$$

$$(by+d)^2=y(ay+c)^2$$

$$b^2y^2+d^2+2bdy=y(a^2y^2+c^2+2acy)$$

$$b^2y^2+d^2+2bdy=a^2y^3+c^2y+2acy^2$$

$$a^2y^3+c^2y+2acy^2-b^2y^2-d^2-2bdy=0$$

$$a^2y^3+(2ac-b^2)y^2+(c^2-2bd)y-d^2=0$$

this is the required equation also roots are

$\alpha^2, \beta^2, \gamma^2$ //

② If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$ then find the equation whose roots are $\frac{\alpha^2-\beta\gamma}{\alpha}, \frac{\beta^2-\alpha\gamma}{\beta}$

Sol: Given that α, β, γ are the roots of the equation.

$$x^3+px^2+qx+r=0 \rightarrow ①$$

$\therefore \alpha+\beta+\gamma = -p$

$\alpha\beta+\alpha\gamma+\beta\gamma = q$

$\alpha\beta\gamma = -r$

To form the equation. chose root. Sate

$$\frac{\alpha^2-\beta\gamma}{\alpha}, \frac{\beta^2-\alpha\gamma}{\beta}, \frac{\gamma^2-\alpha\beta}{\gamma}$$

Let $y = \frac{\alpha^2-\beta\gamma}{\alpha}$

$$= \frac{\alpha}{\alpha}\left(\frac{\alpha^2-\beta\gamma}{\alpha}\right)$$

$$= \frac{\alpha^3-\alpha\beta\gamma}{\alpha^2}$$

$$= \frac{\alpha^3-(-r)}{\alpha^2} = \frac{\alpha^3+\alpha}{\alpha^2}$$

$\therefore y = \frac{x^3+r}{x^2}$ since α is a root of eqn ①

$$x^2y = x^3+r$$

is $x^3-x^2y+r=0 \rightarrow ②$

$$①\Rightarrow \begin{array}{r} x^3+px^2+qx+r=0 \\ \hline -x^2y-px^2-qx=0 \end{array}$$

$\div x$ $\quad -xy-px-q=0$

$\Rightarrow -q = xy+px$

$-q = (y+p)x$

$(y+p)x = -q$

$$\Rightarrow x = \frac{-q}{y+p}$$

using the value of x in eqn ①

$$\left(\frac{-q}{p+y}\right)^3 + p\left(\frac{-q}{p+y}\right)^2 + q\left(\frac{-q}{p+y}\right) + r = 0$$

$$\frac{-q^3}{(p+y)^3} + \frac{pq^2}{(p+y)^2} - \frac{q^2}{p+y} + r = 0$$

$$\frac{-q^3 + pq^2(p+y) - q^2(p+y)^2 + r(p+y)^3}{(p+y)^3} = 0$$

$$-q^3 + p^2q^2 + pq^2y - q^2(p^2 + y^2 + 2py) + r(p^3 + 3p^2y + 3p^2y + y^3) = 0$$

$$-q^3 + p^2q^2 + pq^2y - p^2q^2 - q^2q^2 - q^2y^2 - 2pq^2y + p^3r + 3p^2ry + 3pry^2 + y^3 = 0$$

$$\therefore rq^3 - q^2y^2 + 3prq^2 + pq^2y - 2pq^2y + 3p^2ry - q^{2} + p^3r = 0$$

$$ry^3 + (3pr - q^2)y^2 - pq^2y + 3p^2ry + p^3 + -q^3 = 0$$

this is the required equation

③ If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ then find the equation where root are

$$\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha}, \frac{\alpha}{\gamma}, \frac{\alpha}{\beta} + \frac{\beta}{\gamma}$$

Sol:

given that α, β, γ are the roots of eqn

$$x^3 + qx + r = 0 \rightarrow ①$$

$$\alpha + \beta + \gamma = \frac{-0}{1} = 0 \rightarrow ②$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{q}{1} = q \rightarrow ③$$

$$\alpha\beta\gamma = -\frac{r}{1} = -r \rightarrow ④$$

the from the eqn whose roots are

$$\frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \frac{\gamma}{\beta} + \frac{\beta}{\alpha}$$

let $y = \dfrac{\beta}{\gamma} + \dfrac{\gamma}{\beta}$

$$= \frac{\beta^2 + \gamma^2}{\beta \cdot \gamma}$$

$$= \frac{(\beta+\gamma)^2 - 2\beta\gamma}{\beta\gamma} \quad \left[\begin{array}{l}\text{since } a^2+b^2 \\ = (a+b)^2 - 2ab\end{array}\right]$$

$$= \frac{(-\alpha)^2 - 2\beta\gamma}{\beta\gamma}$$ since eqn ② $\Rightarrow \beta+\gamma = -\alpha$

$$= \frac{\alpha^2 - 2\beta\cdot\gamma}{\beta\cdot\gamma}$$

$$= \frac{\alpha}{\alpha}\left[\frac{\alpha^2 - 2\beta\cdot\gamma}{\beta\cdot\gamma}\right]$$

$$= \frac{\alpha^3 - 2\alpha\cdot\beta\gamma}{\alpha\cdot\beta\cdot\gamma}$$

$$= \frac{\alpha^3 - 2(-r)}{-r}$$ using ④

$$y = \frac{\alpha^3 + 2r}{-r}$$

$$y = \frac{x^3 + 2r}{-r}$$ since α is a roots of eq ①

$$-ry = x^3 + 2r$$

$$x^3 + ry + 2r = 0 \rightarrow ⑤$$

① $\Rightarrow x^3 + qx + r = 0$

⑤ $\Rightarrow x^3 + ry + 2r = 0$

$$qx - ry + r - 2r = 0$$

$$qx - ry - r = 0$$

$$qx = r + ry$$

$$x = \frac{(1+y)r}{q}$$

using the value of γ in eqn① we get

$$\left[\frac{(1+y)r}{q}\right]^3 + q\left[\frac{(1+y)r}{q}\right] + r = 0$$

$$\frac{r^3+(y+1)^3}{q^3} + r + ry + r = 0$$

$$\frac{r^3(y^3+3y^2+3y+1)}{q^3} + 2\gamma + ry = 0$$

$$\frac{r^3y^3+3r^3y^2+3r^3y+r^3+(2r+ry)q}{q^3} = 0$$

$$r^3y^3+3r^3y^2+3r^3y+r^3+2rq^3+ry=0$$

$$r^3y^3+3r^2y^2+(3r^2+rq^3)y+r^3+2rq=0$$

$$= rr^2y^3+3r^3y^2+(3r^2+q^3)y+r^2+2q^3=0$$

which is the required equation.

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4) If α, β, γ are the roots of the eqn $x^3+px+q=0$ find the value of $\left(\frac{1}{\alpha^2}-\frac{1}{\beta\gamma}\right)\left(\frac{1}{\beta^2}-\frac{1}{\alpha\gamma}\right)\left(\frac{1}{\gamma^2}-\frac{1}{\alpha\beta}\right)$

Sol:

Given that α, β, γ are root of the eqn

$$x^3+px+q=0 \to ①$$

$$\alpha+\beta+\gamma = -\frac{0}{1} = 0 \to ②$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{p}{1} = p \to ③$$

$$\alpha\beta\gamma = -\frac{q}{1} = -q \to ④$$

Let us form the eqn whose roots are

$$\frac{1}{\alpha^2}-\frac{1}{\beta\gamma},\ \frac{1}{\beta^2}-\frac{1}{\alpha\gamma},\ \frac{1}{\gamma^2}-\frac{1}{\alpha\beta}$$

Let $y = \frac{1}{\alpha^2} - \frac{1}{\beta\gamma}$

$\frac{1}{\alpha^2} - \frac{\alpha}{\alpha\beta\gamma}$

$= \frac{1}{\alpha^2} - \frac{\alpha}{(-q)}$ using ①

$= \frac{1}{\alpha^2} + \frac{\alpha}{q}$

$= \frac{q+\alpha^3}{\alpha^2 q}$

$y = \frac{q+x^3}{x^2 q}$ since α is are root of eqn ①

$x^2 + yq = x^3 + q$

$x^3 - yqx^2 + q = 0 \rightarrow$ ⑤

① $\Rightarrow x^3 + px + q = 0$

⑤ $\Rightarrow x^3 - yqx^2 + q = 0$

Sub $\quad px + yqx^2 = 0$

$px + yqx^2 = 0$

$\therefore x \cdot p + yqx = 0$

$\Rightarrow x = \frac{-p}{yq}$

using the value of x in eqn ① we get

$\left(\frac{-p}{yq}\right)^3 + p\left(\frac{-p}{yq}\right) + q = 0$

$\frac{-p^3}{y^3 p^3} + \frac{p^2}{yq} + q = 0$

$\frac{-p^3 - p^2(q^2y^2) + q y^3 q^3}{y^3 q^3} = 0$

$$-p^3 - p^2 q^2 y^2 + q^4 y^3 = 0$$

$$q^4 y^3 - p^2 q^2 y^2 - p^3 = 0$$

which is the required eqn, whose roots are

$$\frac{1}{\alpha^2} - \frac{1}{\beta\gamma^2}, \frac{1}{\beta^2} - \frac{1}{\alpha\gamma}, \frac{1}{\gamma^2} - \frac{1}{\alpha\beta}$$

Product of the roots

$$\left(\frac{1}{p\alpha^2} - \frac{1}{\beta\gamma^2}\right)\cdot\left(\frac{1}{\beta^2} - \frac{1}{\alpha\gamma^2}\right)\left(\frac{1}{\alpha^2} - \frac{1}{\alpha\beta}\right) = \frac{-(p^3)}{q^4}$$

$$= \frac{p^2}{q^4}$$

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① If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ then find the equation whose roots are $\beta\gamma - \alpha^2$, $\gamma\alpha - \beta^2$, $\alpha\beta - \gamma^2$

Derive the condition for the roots to be in G.P

Sol:-

Given that α, β, γ are the roots of the eqn

$x^3 + px^2 + qx + r = 0 \rightarrow ①$

$\alpha + \beta + \gamma = -\frac{p}{1} = -p \rightarrow ②$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{q}{1} = q \rightarrow ③$

$\alpha\beta\gamma = -\frac{r}{1} = -r \rightarrow ④$

To from the eqn whose roots are $\beta\gamma - \alpha^2$,

$\gamma\alpha - \beta^2$, $\alpha\beta - \gamma^2$

let $y = \beta\gamma - \alpha^2$

$= \frac{\alpha}{\alpha}\cdot(\beta\gamma - \alpha^2)$

$$= \frac{\alpha\beta\cdot\gamma - \alpha^3}{\alpha}$$

$$y = \frac{-\gamma - \alpha^3}{\alpha}$$

$$y = -\frac{\gamma - x^3}{x}$$ since α is a root of eqn ①

$$= xy = -\gamma - x^3$$

$$⑤ \Rightarrow x^3 + x\cdot y + \gamma = 0$$

$$① \Rightarrow x^3 + px^2 + qx + \gamma = 0$$

$$xy - px^2 - qx = 0$$

$$\div x \quad y - px - q = 0$$

$$-px = q - y$$

$$px = y - q$$

using ⑥ in ① we get

$$\left(\frac{y-q}{p}\right)^3 + p\cdot\left(\frac{y-q}{p}\right)^2 + q\left(\frac{y-q}{p}\right) + \gamma = 0$$

$$\frac{(y-q)^3 + p^2(y-q)^2 + p^2q(y-q) + p^3\gamma}{p^3} = 0$$

$$\Rightarrow (y-q)^3 + p^2(y-q)^2 + p^2q\cdot(y-q) - p^3\cdot\gamma = 0$$

$$y^3 - 3y^2q + 3yq^2 - q^3 + p^2 + y^2 + q^2 - 2yq + p^2qy - p^2q^2 + p^3\gamma = 0$$

$$y^3 - 3y^2q + p^2q^2 + 3q^2 - y - 2p^2qy + p^2qy + q\cdot$$
$$p^2q^2 - p^2q^2 + p^3\gamma = 0$$

$$y^3 + (p - 3q)y^3 + (3q^2 - p^2q) + y + p^3\gamma - q = 0$$

this is the required eqn whose roots are

$\beta\gamma - \alpha^2, \gamma\alpha - \beta^2, \alpha\beta - \gamma^2$

Product of the root is

$$(\beta\gamma-\alpha^2)\cdot(\gamma\alpha-\beta^2)\cdot(\alpha\beta-\gamma^2) = \frac{-p^3-q^3}{1}$$

$$= q^3 - p^3 r \rightarrow ①$$

If the roots are in G.P then we have

$\beta\gamma = \alpha^2$ (or) $\gamma\alpha = \beta^2$ (or) $\alpha\beta = \gamma^2$

$\beta\gamma - \alpha^2 = 0$ (or) $\gamma\alpha - \beta^2 = 0$ (or) $\alpha\beta\gamma^2 = 0$

$\therefore$ eqn ① becomes $q^3 - p^3 r = 0$

This is the conditions for the roots to be in G.P

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② If α, β, γ are the roots of the eqn $x^3 - x - 1 = 0$ then show that $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = -7$

Sol:

Given that α, β, γ are the roots of the equation

$x^3 - x - 1 = 0 \rightarrow$ ⑪

$$\alpha + \beta + \gamma = 0 \rightarrow ②$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-1}{1} = -1 \rightarrow ③$$

$$\alpha\beta\gamma = -\frac{1}{1} = -1 \rightarrow ④$$

To find the eqn whose roots are

$$\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}, \frac{1+\gamma}{1-\gamma}$$

let $y = \frac{1+\alpha}{1-\alpha}$

is $y = \frac{1+x}{1-x}$ since α is a root of eqn

$$(1-x)y = 1+x$$

$$y - xy = 1+x$$

$$y - 1 = x + xy$$

$x + xy = y - 1$

$x(1+y) = y-1$

$x = \frac{y-1}{y+1} \rightarrow ②$

using ② in ① we get

$$\left(\frac{y-1}{y+1}\right)^3 - \left(\frac{y-1}{y+1}\right) - 1 = 0$$

$$\frac{(y-1)^3 - (y-1)\cdot(y+1)^2 - (y+1)^3}{(y+1)^3} = 0$$

is $(y-1)^2 - (y-1)(y+1)^2 - (y+1)^3 = 0$

$y^3 - 3y^2 + 3y - 1 - (y-1)\cdot(y^2+2y+1) -$

$a^3 - 3a^2b + 3ab - b^3$ $(y^3 + 3y^2 + 3y + 1) = 0$

$y^2 - 3y^2 + 3y - 1 - [y^3 + 2y^2 + y - y^2 - 2y - 1] -$

$y^3 - 3y^2 - 3y - 1 = 0$

$y^3 - 3y^2 + 3y - 1 - y^3 - 2y^2 - y + y^2 + 2y + 1$

$- y^3 - 3y^2 - 3y - 1 = 0$

$-y^3 - 7y + y - 1 = 0$

$\times (-1)$

$y^3 + 7y^2 - y + 1 = 0$

which is the required eqn whos roots are

$\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}, \frac{1+\gamma}{1-\gamma}$

$\therefore$ Sum of the roots is $\frac{1+\alpha}{1-\alpha}$

$$\frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = \frac{-7}{1}$$
$$= -7$$

Show that the equation $x^7-3x^4+2x^3-1=0$ has atleast 4 imaginary roots

Sol:

Given equation is $f(x)=x^7-3x^4+2x^3+1=0$

sign of the term of $f(x)$ are

$+ \; - \; + \; -$

there are 3 changes of sign of the term of $f(x)$

$\therefore f(x)=0$ has at most 3

positive real roots

or $f(-x) = -x^7 - 3(-x)^4 + 2(-4)^3$

$= -x^7 - 3x^4 - 2x^3 - 1$

sign of the terms of $f(-x)$ are

$- \; - \; - \; -$

there is no changes in the sign of the term of $f(-x)$

$\therefore$ There is no negative real roots

$\therefore$ there can be almost 3 real roots But $f(x)$ as of degree 7

$\therefore f(x)$ has 7 roots

Imaginary roots always occur in pairs

$\therefore f(x)=0$ has atlest 4 imaginary roots.

(2) Shows that the equation $x^4+3x-1=0$ has 2 real and 2 imaginary roots.

Sol:

Let $f(x)$ $x^4+3x-1=0$

Signs of the terms of $f(x)$ are

+ + −

There is only one change of the terms of $f(x)$

$\therefore f(x)=0$ has alway 1 positive real root

Now $f(-x)$ = $-x^4+3(-x)-1$

$\therefore$ Signs of the terms of $f(-x)$ are

+ − −

there is only one change of sign of the term of $f(-x)$

$\therefore f(x)=0$ has almost 1 negative real root

$f(0)=0+0-1=-1<0$

$f(1)=1+3-1=3>0$

$\therefore$ There is a root between 0 and 1 But $f(x)$ in of degree 4

$\therefore f(x)=0$ has 4 roots

But imaginary roots occur in pairs

$\therefore f(x)=0$ has 2 real and 2 imaginary roots

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3. Show that the equation $3x^5-2x^3-4x+2=0$ has 3 real roots and 2 imaginary roots.

Sol:

Given equation is $f(x)=3x^5-2x^3-4x+2=0$

sign of the term of $f(x)$ are

$+ \quad - \quad - \quad +$

There are 2 changes of sing of the term $f(x)$

$f(x)=0$ has almost 2 positive real roots

$f(-x) = 3(-x)^5 - 2(-x)^3 - 4(-x) + 2 = 0$

$f(-x) = -3x^5 + 2x^3 + 4x + 2 = 0$

$\therefore$ signs of the term of $f(-x)$ are

$- \quad + \quad + \quad +$

there is only one changes of singns of the terms of $f(-x)$

$f(x)=0$ has almost 1 negative root

$f(0) = 0 + 0 + 0 + 2 = 2 > 0$

$f(1) = +3 - 2 - 4 + 2 = -1 < 0$

$f(2) = 96 - 16 + 2 = 74 > 0$

$\therefore$ There is a root between 0 and 1 and another root between 1 and 2

$\therefore$ There are exactly 2 postive ~~root ro~~ real root but $f(x)=0$ is of degree 5

$\therefore$ It has 5 roots

But Imaginary root occur in pairs

$\therefore$ $f(x)=0$ has 2 imaginary roots

$\therefore$ $f(x)=0$ has 1 negative roots

$\therefore$ $f(x)=0$ has 3 real roots and 2 imaginary roots

5. Shows that the equation $x^9+x^5+x^4+x^2+1$ has one real root which is negative and 8 imaginary roots

sol:-

Given equation is $f(x) = x^9+x^5+x^4+x^2+1=0$

signs of the term of $f(x)$ are

$+ + + +$

there are no changes of the singns of the term

$f(-x) = (-x)^9 + (-x^5) + (-x)^4 + (-x^2) + 1$

$= -x^9 - x^5 + x^4 + x^2 + 1$

$\therefore$ signs of the term of $f(-x)$ are

$- - + +$

there is one changing of the sign of the term of $f(-x)$

There can be almost 1 Negative root

$f(0) = 0+0+0+0+1 = 1>0$

$f(1) = 1+1+1+1+1 = 5>0$

there is a root almost 8 negative roots and almost 8 Imaginary roots of pairs.