19UMAA01

MATRICES

B.SC. MATHEMATICS

I - SEMESTER.

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unit - II

PCF: characteristic equation

let A be a square matrix of order n and λ be a Scalar. let I be the unit matrix of order n. then the equation let $|A-\lambda I|$ is called the characteristic equation of the matrix A

cayley Hamilton theorom

Statement: Every square matrix satisfies its own characteristic equation } ①

I Find the characteristic equation of the following matrices.

① $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$ ② $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Sol:

① Given matrix is $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$

the characteristic equation of A is

$\lambda^2 - S_1\lambda + S_2 = 0 \rightarrow ①$

where S_1 = Sum of the principal diagonal elements

$= 5+3$

$= 8$

$S_1 = 8$

$S_2 = \det|A| = \begin{vmatrix} 5 & 3 \\ 1 & 3 \end{vmatrix} = 15 - 3 = 12$ $S_2 = 12$

∴ From eqn ①, the char. eqn of A is

$\lambda^2 - 8\lambda + 12 = 0$

② Give matrix is $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

the characteristic equation of A is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \rightarrow ①$$

where S_1 = Sum of the main diagonal elements

$= 2+3+2$

$S_1 = 7$

S_2 = Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$$

$= (6-2) + (4-1) + (6-2)$

$= 4+3+4$

$S_2 = 11$

$S_3 = \det |A|$

$$= \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$= 2[6-2] - 2\{2-1\} + 1[2-3]$

$= 2(4) - 2(1) + 1(1)$

$= 8-2-1$

$S_3 = 5$

$\therefore$ From eqn ①, the char. eqn of A is

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

2. Verify cayley hamilton thorem for the matrix $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$ and hance find A^{-1} and A^3

Sol: Given matrix is $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$

The char eqn of A is $\lambda^2 - S_1\lambda + S_2 = 0$

cayley Hamilton theorem States that

$$A^2 - 3A - 10I = 0 \longrightarrow (1)$$

which is to be verified

$$A^2 = A \times A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 7 & 9 \\ 6 & 22 \end{pmatrix}$$

$$\therefore A^2 - 3A - 10I = \begin{pmatrix} 7 & 9 \\ 6 & 22 \end{pmatrix} - 3\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 9 \\ 6 & 22 \end{pmatrix} + \begin{pmatrix} 3 & -9 \\ -6 & -12 \end{pmatrix} + \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} 7+3-10 & 9-9+0 \\ 6-6+0 & 22-12-10 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore A^2 - 3A - 10I = 0$$

$\therefore$ caley Hamilton theorem is verified

To find A^{-1}

From eqn (1), we have

$$A^2 - 3A - 10I = 0$$

premultiply by A^{-1} on the sides we get

$$A^{-1} \cdot (A^2 - 3A - 10I) = A^{-1} 0$$

$$A^{-1}A^2 - 3A^{-1}A - 10A^{-1}I = 0$$

$$A - 3I - 10A^{-1} = 0$$

$$A - 3I = 10A^{-1}$$

$$10A^{-1} = A - 3I$$

$$= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

$$10A^{-1} = \begin{pmatrix} -4 & 3 \\ 2 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{10}\begin{pmatrix} -4 & 3 \\ 2 & 1 \end{pmatrix}$$

To find A^3

From eqn ①, we have $A^2 - 3A - 10I = 0$ premultiply by A on both sides we get

$$A(A^2 - 3A - 10I) = A0$$

$$A^3 - 3A^2 - 10A = 0$$

$$\Rightarrow A^3 = 3A^2 + 10A$$

$$= 3\begin{pmatrix} 7 & 9 \\ 6 & 22 \end{pmatrix} + 10\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 27 \\ 18 & 66 \end{pmatrix} + \begin{pmatrix} -10 & 30 \\ 20 & 40 \end{pmatrix}$$

$$\therefore A^3 = \begin{pmatrix} 11 & 57 \\ 38 & 106 \end{pmatrix}$$

Verify cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ and also use it to find A^{-1} and A^4

sol:

Given matrix is $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

the char eqn of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \rightarrow ①$

where S_1 = Sum of the main diagonal elements

$$S_1 = 1 + 2 + 1$$

$$S_1 = 4$$

S_2 = Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

$$= (2-6) + (1-7) + (2-12)$$

$$= -4 - 6 - 10$$

$$S_2 = -20$$

$$S_3 = \det|A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 1(2-6) - 3(4-3) + 7(8-2)$$

$$= -4 - 3 + 42$$

$$S_3 = 35$$

$\therefore$ the char. eqn of A is from eqn ① is

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

Verification of cayley Hamilton theorem is to verify: $A^3 - 4A^2 - 20A - 35I = 0 \rightarrow$ ②

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$A^3 = A \cdot A^2 = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$\therefore A^3 = \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix}$$

$$\therefore A^3 - 4A^2 - 20A - 35I$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} - 4\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - 20\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - 35\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 11 \end{pmatrix} + \begin{pmatrix} -80 & -92 & -92 \\ -60 & -88 & -148 \\ -40 & -36 & -56 \end{pmatrix} + \begin{pmatrix} -30 & -60 & -140 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{pmatrix}$$

$$+\begin{pmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{pmatrix}$$

$$= \begin{pmatrix} 135-80-20+35 & 152-92-60+0 & 232-92-140+0 \\ 140-60-80 & 163-88-40+35 & 208-148-60+0 \\ 60-36-40+0 & 76-36-40+0 & 111-56-20+35 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore A^3 - 4A^2 - 20A - 35I = 0$

Hence cayley Hamilton theorem is verified

to find A^{-1}

From eqn ②, we have

$$A^3 - 4A^2 - 20A - 35I = 0$$

premultiplying by A^{-1} on both sides we get

$$A^{-1}(A^3 - 4A^2 - 20A - 35I) = A^{-1}0$$

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

$$\Rightarrow 35A^{-1} = A^2 - 4A - 20I$$

$$35A^{-1} = \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - 4\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - 20\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 20-4-20 & 23-12+0 & 23-28+0 \\ 15-16-0 & 22-8-20 & 37-12+0 \\ 10-4+0 & 9-8+0 & 14-4-20 \end{pmatrix}$$

$$35A^{-1} = \begin{pmatrix} -4 & 11 & -5 \\ -1 & 6 & 25 \\ 6 & 1 & -10 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{35}\begin{pmatrix} -4 & +11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{pmatrix}$$

To find A^4

From eqn ⑤, we have $A^3 - 4A^2 - 20A - 35I = 0$

premultiplying by A on bothsides we get

$$A^4 - 4A^3 - 20A^2 - 35A = 0$$

$$\Rightarrow A^4 = 4A^3 + 20A^2 + 35A$$

$$= 4\begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} + 20\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$+ 35\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 540 & 608 & 928 \\ 560 & 652 & 832 \\ 180 & 76 & 444 \end{pmatrix} + \begin{pmatrix} 400 & 460 & 460 \\ 300 & 440 & 740 \\ 200 & 180 & 280 \end{pmatrix}$$

$$+ \begin{pmatrix} 35 & 105 & 245 \\ 140 & 70 & 105 \\ 35 & 70 & 35 \end{pmatrix}$$

$$= \begin{pmatrix} 540+400+35 & 608+460+105 & 928+460+245 \\ 560+300+140 & 652+440+70 & 832+740+105 \\ 180+200+35 & 76+180+70 & 444+280+35 \end{pmatrix}$$

$$= \begin{pmatrix} 975 & 1173 & 1633 \\ 1000 & 1162 & 1677 \\ 415 & 326 & 759 \end{pmatrix}$$

Verify cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$ and hence find A^{-1} and A^4

Given matrix is $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

The char eqn of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ —→ ①

where S_1 = Sum of the main diagonal elements

$= 1+2+3$

$S_1 = 6$

S_2 = Sum of the main of the main diagonals elements

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-2) + (3+2) + (2-0)$$

$$= 4+5+2$$

$S_2 = 11$

$$S_3 = \det |A| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1(6-2) - 0(3-2) - 1(2-4)$$

$$= 1(4) - 0(1) - 1(-2)$$

$$= 4+2$$

$S_3 = 6$

the char eqn of A is. from eqn ① is.

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Verification of cayley Hamiltion theorem is to verify

$A^3 - 6A^2 + 11A - 6I = 0$

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0-2 & 0+0-2 & -1+0-3 \\ 1+2+2 & 0+4+2 & -1+2+3 \\ 2+2+6 & 0+4+6 & -2+2+9 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 & -4 \\ 5 & 6 & 4 \\ 10 & 10 & 9 \end{pmatrix}$$

$$A^3 = AA^2 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & -2 & -4 \\ 5 & 6 & 4 \\ 10 & 10 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} -1+0-10 & -2+0-10 & -4+0-9 \\ -1+10+10 & -2+12+10 & -4+8+9 \\ -2+10+30 & -4+12+30 & -8+8+27 \end{pmatrix}$$

$$= \begin{pmatrix} -11 & -12 & -13 \\ 19 & 20 & 13 \\ 38 & 38 & 27 \end{pmatrix}$$

$\therefore A^3 - 6A^2 + 11A - 6I$

$$\begin{pmatrix} -11 & -12 & -13 \\ 19 & 20 & 13 \\ 38 & 38 & 27 \end{pmatrix} - 6\begin{pmatrix} -1 & -2 & -4 \\ 5 & 6 & 4 \\ 10 & 10 & 9 \end{pmatrix} + 11\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -11 & -12 & -13 \\ 19 & 20 & 13 \\ 38 & 38 & 27 \end{pmatrix} + \begin{pmatrix} 6 & 12 & 24 \\ -30 & -36 & -24 \\ -60 & -60 & -54 \end{pmatrix} + \begin{pmatrix} 11 & 0 & -11 \\ 11 & 22 & 11 \\ 22 & 22 & 33 \end{pmatrix} + \begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} -11+6+11-6 & -12+12+0+0 & -13+24+11+0 \\ 19-30+11+0 & 20-36+22-6 & 13-24+11+0 \\ 38-60+22+0 & 38-60+22+0 & 27-54+33-6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore A^3 - 6A^2 + 11A - 6I = 0$

Hence cayley Hamiltion theorem is verified to find A^{-1}

From eqn (2), we have

$A^3 - 6A^2 + 11A - 6I = 0$

premutiplying by A^{-1} on bot hindes we get

$A^{-1}(A^3 - 6A^2 + 11A - 6I) = 0$

$A^2 - 6A + 11I - 6A^{-1} = 0$

$6A^{-1} = \cancel{6A+1} \; A^2 - 6A + 11I$

$$6A^{-1} = \begin{pmatrix} -1 & -2 & -4 \\ 5 & 6 & 4 \\ 10 & 10 & 9 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} + 11\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$6A^{-1} = \begin{pmatrix} -1 & -2 & -4 \\ 5 & 6 & 4 \\ 10 & 10 & 9 \end{pmatrix} + \begin{pmatrix} -6 & 0 & 6 \\ -6 & -12 & -6 \\ -12 & -12 & -18 \end{pmatrix} + \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} -1-6+11 & -2+0+0 & -4+6+0 \\ 5-6+0 & 6-12+11 & 9-6+0 \\ 10-12+0 & 10+2+0 & 9-18+11 \end{pmatrix}$$

$$6A^{-1} = \begin{pmatrix} 4 & -2 & +2 \\ -1 & 5 & -2 \\ -2 & -2 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 4 & -2 & 2 \\ -1 & 5 & -2 \\ -2 & -2 & -1 \end{pmatrix}$$

To find A^4

From eqn ②, we have

$$A^3 - 6A^2 + 11A - 6I = 0$$

$$A(A^3 - 6A^2 + 11A - 6I) = 0$$

$$A^4 - 6A^3 + 11A^2 - 6A = 0$$

$$\Rightarrow A^4 = 6A^3 - 11A^2 - 11A^2 + 6A$$

$$= 6\begin{pmatrix} -11 & -12 & -13 \\ 19 & 20 & 13 \\ 38 & 38 & 38 \end{pmatrix} - 11\begin{pmatrix} -1 & -2 & -4 \\ 5 & 6 & 4 \\ 10 & 10 & 9 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -66 & -72 & -78 \\ 114 & 120 & 78 \\ 228 & 228 & 228 \end{pmatrix} + \begin{pmatrix} 11 & 22 & 44 \\ -55 & -66 & -44 \\ -110 & -110 & -99 \end{pmatrix} + \begin{pmatrix} 6 & 0 & -6 \\ 6 & 12 & 6 \\ 12 & 12 & 18 \end{pmatrix}$$

$$\begin{pmatrix} -66+11+6 & -72+22+0 & -78+44-6 \\ 114-55+6 & 120-66+12 & 78-44+6 \\ 228-110+12 & 228-110+12 & 228-99-108 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{pmatrix}$$

Eigen Values and Eigen Vectors.

Let $A = [a_{ij}]$ be a square matrix of order n If there exists a non-zero column vector x and a scalar λ such that $Ax = \lambda x$. then λ is called an eigen value of the matrix A and x is called the eigen vector corresponding to the eigen value λ

① Find the eigen value and the eigen vectors of the matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

Sol: Given matrix is $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

the char eqn of A is $\lambda^2 - S_1\lambda + S_2 = 0$

where S_1 = sum of the main diagonal elements

$= 5+2$

$S_1 = 7$

$S_2 = \det|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10-4 = 6$

$S_2 = 6$

∴ the char eqn of A is $\lambda^2 - 7\lambda + 6 = 0$

$(\lambda-1)(\lambda-6) = 0$

$\Rightarrow \lambda = (1, 6)$

∴ The eigen values of A are 1 and 6 to find the eigen vector

let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector consider the eqn $(A - \lambda I)\lambda = 0 \rightarrow ①$

case ① let $\lambda = 1$

∴ the eigen vector corresponding to $\lambda = 1$ is from eqn ①, we have

$$\begin{pmatrix} 5-1 & 4 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 4x_1 + 4x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 + x_2 = 0$$

$x_1 = -x_2$

$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1}$

$\therefore$ the eigen vector corr to $\lambda = 1$ is

$$X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

case 2

let $\lambda = 6$

$\therefore$ the eigen vector corresponding to $\lambda = 6$ is

from eqn ①, we have

$$\begin{pmatrix} 5-6 & 4 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -x + 4x_2 = 0 \\ x_1 - 4x_2 = 0 \end{array}\right] \Rightarrow x_1 - 4x_2 = 0$$

$$\Rightarrow x_1 = 4x_2$$

$$\frac{x_1}{4} = \frac{x_2}{1}$$

$\therefore$ the eigen vector corr to the eigen value $\lambda = 6$ is

$$x_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Eigen value	Eigen vector
$\lambda = 1$	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
$\lambda = 6$	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

② Find the characteristic roots of the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Sol: det $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

the char eqn of A is $\lambda^2 - S_1\lambda + S_2 = 0$

where S_1 = sum of the main diagonal elements

$= \cos\theta + \cos\theta$

$S_1 = 2\cos\theta$

$S_2 = \det|A| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta)$

$= \cos^2\theta + \sin^2\theta$

$= 1$

∴ the char eqn of A is

$\lambda^2 - (2\cos\theta)\lambda + 1 = 0$ $\quad ax^2 - bx + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \lambda = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4(1)(1)}}{2 \times 1}$

$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$ $\quad \sin^2\theta + \cos^2\theta$

$= \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}}{2}$ $\quad \Rightarrow \cos^2\theta - 1 = -\sin\theta$

$= \frac{2\cos\theta \pm \sqrt{4(-\sin\theta)}}{2}$

$= \frac{2\cos\theta \pm \sqrt{-4\sin^2\theta}}{2}$

$= 2\cos\theta \pm \sqrt{4 i^2 \sin^2\theta}$

$= \frac{2\cos\theta \pm 2i\sin\theta}{2}$

$$= \frac{2(\cos\theta \pm i\sin\theta)}{2}$$

$$= \cos\theta \pm i\sin\theta$$

$\therefore$ the char roots are $\cos\theta + i\sin\theta$, $\cos\theta - i\sin\theta$

3. Find the eigen vectors of the matrix $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

Sol: $\det A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

the char eqn of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ — (1)

where S_1 = Sum of the main diagonal elements

$= 3+5+3$

$S_1 = 11$

S_2 = Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$= (15-1) + (9-1) + (15-1)$

$= 14 + 8 + 14$

$S_2 = 36$

$S_3 = \det|A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(15-1) + 1(-3+1) + 1(1-5)$

$= 3(14) - 2 - 4$

$= 42 - 6$

$S_3 = 36$

$\therefore$ from eqn ① the char eqn of A is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

here $\lambda = 2$ is a root

$\therefore$ by synthetic division, we get

$$\begin{array}{c|cccc} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\Rightarrow \lambda^2 - 9\lambda + 18 = 0$$

$$(\lambda - 3)(\lambda - 6) = 0$$

$$\Rightarrow \lambda = 3, 6$$

$\therefore$ the eigen values of A are 2, 3, 6

the find the eigen vector

let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector corrseder the eqn

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow ②$$

case ① let $\lambda = 2$

$\therefore$ eqn ② given

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

case ① let $\lambda = 2$

$\therefore$ eqn ② given

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 - x_2 + x_3 = 0 \quad \Rightarrow 2-1-1=0$

$-x_1 + 3x_2 + x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$\Rightarrow -x_1 + 3x_2 + x_3 = 0$

$\therefore$ By the method of cross multiplication, we get

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_1 & x_2 \\ \hline 1 & -1 & 1 & 1 & -1 \\ -1 & 3 & 1 & -1 & 3 \end{array}$$

$$\frac{x_1}{-1-3} = \frac{x_2}{-1-1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{2}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

the eigen vector corr to $\lambda = 2$ is

$$X_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

case ② let $\lambda = 3$

$\therefore$ eqn ① gives

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$0x_1 - x_2 + x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

$x_1 - x_2 + 0x_3 = 0$

Taking the first and last eqn and using the method of cross multiplication, we get

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_1 & x_2 \\ \hline 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & 1 \end{array}$$

$$\frac{x_1}{0+1} = \frac{x_2}{1-0} = \frac{x_3}{0+1}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ the eigen vector corr to $\lambda = 3$ is $x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

case ③ let $\lambda = 6$

$\therefore$ eqn ③ given

$$\begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - 3x_3 = 0$$

Taking the first and two eqn and using the method of cross multiplication, we get

x_1	x_2	x_3	x_1	x_2
0	-1	1	0	-1
1	-1	0	1	1

$$\frac{x_1}{0+1} = \frac{x_2}{1-0} = \frac{x_3}{0+1}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ the eigen vector corr to $\lambda = 3$ is

$$x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

case③. let $\lambda = 6$

$\therefore$ eqn ③ given

$$\begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - 3x_3 = 0$$

Taking the first and two eqn and using the method of cross multiplication we get,

x_1	x_2	x_3	x_1	x_2
-3	-1	1	-3	-1
-1	-1	-1	-1	-1

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore$ the eigen vector corr to $\lambda = 6$ is

$x_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Eigen value	Eigen vector
$\lambda = 2$	$x_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$
$\lambda = 3$	$x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\lambda = 6$	$x_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Find the eigen vectors of the matrix

① $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ ② $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$

① $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Given matrix is $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

the char eqn of A is $A = \lambda^2 - S_1\lambda + S_2 = 0$ —→①

wher S_1 = sum by the main diagonal elements

$= 4+2$

$= 6$

$S_2 = |A| = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 8-3$

$= 5$

$S_2 = 5$

$\therefore$ the char eqn of A is. $\lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 1)(\lambda - 5) = 0$

$\therefore$ the eigen values of A are and 5

to find the eigen vector

let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector

consider the eqn $(A - \lambda I)x = 0 \rightarrow ②$

case ①

let $A = 1$

$\therefore$ the eigen vector corresponding to $\lambda - 1$ is form eqn ②, we have

$$\begin{pmatrix} 4-1 & 1 \\ 3 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} 3x_1 + x_2 = 0 \\ 3x_1 + x_2 = 0 \end{matrix}\right\} 3x_1 + x_2 = 0$$

$$3x_1 = -x_2$$

$$S_2 = \cancel{\det(A)} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 8 - 3 = 5$$

$$\frac{x_1}{-1} = \frac{x_2}{3}$$

$\therefore$ the eigen Vector corr to $\lambda = 1$ is

$$x_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

case ② $\lambda = 5$

let $\lambda = 5$

$\therefore$ the eigen Vector corresponding to $\lambda = 5$ is form eqn ①, we have

$$\begin{pmatrix} 4-5 & 1 \\ 3 & 2-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} -x_1 + x_2 \\ 3x_1 - 3x_2 \end{matrix}\right\} x_1 + x_2 = 0$$

$-x_1 = -x_2$

$$\frac{x_1}{-1} = \frac{x_2}{-1}$$

∴ the eigen vector corr to $\lambda = 5$ is

$$x_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Eigen value	Eigen vector
$\lambda = 1$	$x_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$
$\lambda = 5$	$x_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$

Given matrix is $A = \begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$

the char eqn of A is $\lambda^2 - S_1\lambda + S_2 = 0$

wher S_1 = Sum of the main diagonal elements

$S_1 = 1 + 4$

$S_1 = 5$

$S_2 = \det |A| = \begin{vmatrix} 1 & -2 \\ 5 & 4 \end{vmatrix}$

$= 4 - 10$

$S_2 = -6$

∴ the char eqn of A is $\lambda^2 - 5\lambda - 6 = 0$

$(\lambda + 1)\cdot(\lambda - 6) = 0$

∴ the eigen values of A are -1 and 6

To find the eigen vector

let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector

consider the eqn $(A - \lambda I)x = 0 \rightarrow (2)$

case (1) let $\lambda = -1$

$$\begin{pmatrix} 1+1 & -2 \\ -5 & 4+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 = 0$$

$$\left.\begin{array}{l} 2x_1 - 2x_2 = 0 \\ -5x_1 + 5x_2 = 0 \end{array}\right\}$$

$$-x_1 + x_2 = 0$$

$$2x_1 = 5x_2$$

$$\frac{x_1}{5} = \frac{x_2}{2}$$

$\therefore$ the eigen vector corresponding to the eigen value $\lambda = -1$ is $X_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

case ②

let $\lambda = 6$

$$\begin{pmatrix} 1-6 & -2 \\ -5 & 4-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & -2 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -5x_1 - 2x_2 = 0 \\ -5x_1 - 2x_2 = 0 \end{array}\right\} -5x_1 - 2x_2 = 0$$

$$-5x_1 - 2x_2 = 0$$

$$-5x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{-5}$$

$\therefore$ the eigen Vector corr. to $\lambda = 6$ is $X_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

Eigen value	Eigen vector
$\lambda = -1$	$X_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$
$\lambda = 6$	$X_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

Find all the eigen vectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

sol:

Given matrix is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

the char eqn of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

when S_1 = Sum of the main diagonal elements

$= 0 + 0 + 0$

$S_1 = 0$

S_2 = Sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= (0-1) + (0-1) + (0-1)$

$= -1-1-1$

$= -3$

$$S_3 = \det|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$= 0(0-1) - 1(0-1) + 1(1-0)$

$= 0(-1) - 1(-1) + 1(1)$

$= 0 + 1 + 1$

$= 2$

The char eqn of A is $= \lambda^3 - 3\lambda - 2 = 0$

Here $\lambda = -1$ is a root

$\therefore$ By Sythetic division, we get

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\Rightarrow \lambda^2 - \lambda - 2 = 0$

$(\lambda + 1)(\lambda - 2) = 0$

$\Rightarrow \lambda = -1, 2$

$\therefore$ the eigen values are $-1, -1, 2$

To find the eigen vector

let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector consider the eqn

$(A-\lambda I)X = 0$

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (1)$$

case (1)

$\lambda = -1$

$\therefore$ eqn (1) given

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 0 \rightarrow (2)$

put $x_1 = 0$ in eqn (2), we get

$x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$

$\Rightarrow \frac{x_1}{-1} = \frac{x_3}{1}$

$\therefore X_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

put $x_2 = 0$ in eqn (2), we get

$x_1 + x_3 = 0$

$\Rightarrow x_1 = -x_3$

$\Rightarrow \frac{x_1}{-1} = \frac{x_3}{1}$

$X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

case (2) let $\lambda = 2$

$\therefore$ Eqn (1) gives

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0$$
$$x_1 - 2x_3 + x_3 = 0$$
$$x_1 + x_2 - 2x_3 = 0$$

Taking the first and the second eqn and using the method of cross multiplication we get.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_1 & x_2 \\ \hline -2 & 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 & -2 \end{array}$$

$$\therefore \frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

ie $\dfrac{x_1}{3} = \dfrac{x_2}{3} = \dfrac{x_3}{3}$

ie $\dfrac{x_1}{1} = \dfrac{x_2}{1} = \dfrac{x_3}{1}$

The eigen vector corr to the eigen value is $\lambda = 2$

$$= X_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Eigen value	Eigen Vector
$\lambda = -1$	$X_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$
$\lambda = -1$	$X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
$\lambda = 2$	$X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Find eigen values eigen vectors of the matrix

$$A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$$

Sol :

Given matrix is $A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$

The char eqn of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3$ where

S_1 = Sum of the main diagonal elements

$= 11 - 2 - 6$

$S_1 = 3$

S_2 = Sum of the minors of the main diagonal elements

$= \begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -7 \\ 10 & -6 \end{vmatrix} = \begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix}$

$= (12 - 20) + (-66 + 70) + (-22 + 28)$

$= -8 + 4 + 6$

$S_2 = 2$

$S_3 = \det|A| = \begin{vmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{vmatrix}$

$= 11(12 - 20) + 4(-42 + 50) - 7(-20 + 20)$

$= 11(-8) + 4(8) - 7(-8)$

$= -88 + 32 + 56$

$S_3 = 0$

$\therefore$ from eqn ①, the char eqn or A is

$\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\Rightarrow \lambda = 0$ (or) $\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\Rightarrow \lambda = 1, 2$

$\therefore$ the eigen values of A are 0, 1, 2 to find the eigen Vector

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector consider

the eqn $(A-\lambda I)X = 0$

$$\begin{pmatrix} 11-\lambda & -4 & -7 \\ 2 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

case (1) let $\lambda = 0$

$$\begin{pmatrix} 11-0 & -4 & -7 \\ 7 & -2-0 & -5 \\ 10 & -4 & -6-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$11x_1 - 4x_2 - 7x_3$

$7x_1 - 2x_2 - 5x_3$

$10x_1 - 4x_2 - 6x_3$

Taking the first and second eqn and using the method of cross multiplication we get

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_1 & x_2 \\ \hline 11 & -4 & -7 & 11 & -4 \\ 7 & -2 & -5 & 7 & -2 \end{array}$$

$$\frac{x_1}{20-14} = \frac{x_2}{-49+55} = \frac{x_3}{-22+28}$$

$$\frac{x_1}{6} = \frac{x_2}{6} = \frac{x_3}{6}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

the eigen vector corr to the eigen value of $\lambda = 0$ is $\Big\}$ $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

case (2) let $\lambda = 1$

$$\begin{pmatrix} 4-1 & -4 & -7 \\ 7 & -2-1 & -5 \\ 10 & -4 & -6-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 10x_1 - 4x_2 - 7x_3 &= 0 \\ 7x_1 - 3x_2 - 5x_3 &= 0 \\ 10x_1 - 4x_2 - 7x_3 &= 0 \end{aligned}\right\} 10x_1 - 4x_2 - 7x_3$$

$10x_1 - 4x_2 - 7x_3$

$7x_1 - 3x_2 - 5x_3$

∴ By the method of cross multiplication we get

x_1	x_2	x_3	x_1	x_2
10	-4	-7	10	-4
7	-3	-5	7	-3

$$\frac{x_1}{20-21} = \frac{x_2}{-49+50} = \frac{x_3}{-30+28}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-2}$$

the eigenvector corr to the eigen value of $\lambda = 1$

is $x_2 \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

case ③ Let $\lambda = 2$

$$\begin{pmatrix} 1-2 & -4 & -7 \\ 7 & -2-2 & -5 \\ 10 & -4 & -6-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$9x_1 - 4x_2 - 7x_3 = 0$$

$$7x_1 - 4x_2 - 5x_3 = 0$$

$$10x_1 - 4x_2 - 8x_3 = 0$$

Taking the first and second eqn and using the method of cross multiplication we get.

x_1	x_2	x_3	x_1	x_2
9	-4	-7	9	-4
7	-4	-5	7	-4

$$\frac{x_1}{20-28} = \frac{x_2}{-49+45} = \frac{x_3}{-36+20}$$

$$\frac{x_1}{-8} = \frac{x_2}{-4} = \frac{x_3}{-8}$$

$$-4 \div \quad \frac{x_1}{+2} = \frac{x_2}{+1} = \frac{x_3}{+2}$$

$\therefore$ the eigen vector corr. to the eigen value of $\lambda = 2$ is $\Big\}\; x_3 = \begin{pmatrix} +2 \\ +1 \\ +2 \end{pmatrix}$

eigen value	eigen vector
$\lambda = 0$	$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\lambda = 1$	$x_2 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$
$\lambda = 2$	$\lambda_3 = \begin{pmatrix} +2 \\ +1 \\ +2 \end{pmatrix}$

Find all the eigen vectors of the matrin $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

sol:

Given matrix is $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

the given matrix is A is an upper triangulal matrix

$\therefore$ the eigen values of A is are the elements of the main diagonal

$\therefore$ The eigen values of A are 2, 2, 2 to find the eigen vector

let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector consider the eqn $(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

put $\lambda = 2$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_2 = 0$

$x_3 = 0$ and x_1 is arbitary

$\therefore$ let $x_1 = k$ (say)

where k is any arbitary constant

$$\therefore X = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ Eigen value	eigen vector
2, 2, 2	$X = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$ wher k is any arbitary constant