19UMA02

DIFFERENTIAL CALCULUS

B.Sc. MATHEMATICS

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NAMAKKAL (DT).

UNIT-II

JACOBIANS,
MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES,
NECESSARY AND SUFFICIENT CONDITIONS,
METHOD OF LAGRANGE'S MULTIPLIERS,
SIMPLE PROBLEMS,

UNIT-2

Jacobians:

Definition:

If $u_1, u_2, \ldots$ are functions of 'n' variables $x_1, x_2, \ldots x_n$. Then Jacobians of Transformation. term $x_1, x_2, \ldots x_n$ to $u_1, u_2, \ldots u_n$ is defined by determinant.

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1}, & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1}, & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_n}{\partial x_1}, & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

And is denoted by the symbol.

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2 \ldots x_n)} \text{ (or) } J(u_1, u_2, \ldots u_n).$$

In Particular $\dfrac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

1. If $x = u(1+v)$ and $y = v(1+u)$ Find $\frac{\partial(x,y)}{\partial(u,v)}$.

Solution :-

$$\frac{\partial x}{\partial u} = (1+v) \qquad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1+u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} (1+v) & u \\ v & (1+u) \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1+u+v+\cancel{uv}-\cancel{uv}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = 1+u+v$$

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2. If $x+y+z=u$, $y+z=uv$, $z=uvw$ Prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.

Solution :-

$x+y+z=u$

$x = -y-z+u$

$x = -(y+z)+u$

$\boxed{x = -uv+u}$

$y+z=uv$

$y = uv - z$

$\boxed{y = uv - uvw}$

$\boxed{z = uvw}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$\frac{\partial x}{\partial u} = 1-v \quad \frac{\partial x}{\partial v} = -u \quad \frac{\partial x}{\partial w} = 0$

$\frac{\partial y}{\partial u} = v - vw \quad \frac{\partial y}{\partial v} = u - uw \quad \frac{\partial y}{\partial w} = -uv$

$\frac{\partial z}{\partial u} = vw \quad \frac{\partial z}{\partial v} = uw \quad \frac{\partial z}{\partial w} = uv$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

(Direct Determent painneum podalam)

(or)

Eeppadum podalam)

$$\begin{array}{c} \\ = \end{array}\begin{array}{ccc} & & C_3 \\ \end{array}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} R_2 \to R_2 + R_3$$

Taking C_3 because easy to simplify

$$= 0(\,\cdot\,) - 0(\;) + u^2v(1 - \cancel{v} + \cancel{v})$$

$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$ is proved.

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2). If $y_1 = \cos x_1,\ y_2 = \sin x_1 \cos x_2,\ y_3 = \sin x_1 \sin x_2 \cos x_3$

$y_n = \sin x_1 \cdot \sin x_2, \dots \sin x_{n-1} \cos x$

find $\frac{\partial(y_1, y_2 \dots y_n)}{\partial(x_1, x_2, \dots x_n)}$

Solution:

$$J = \frac{\partial(y_1 \dots y_n)}{\partial(x_1, x_2 \dots x_n)}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$\frac{\partial y_1}{\partial x_1} = -\sin x_1 \qquad \frac{\partial y_1}{\partial x_2} = 0 \qquad \frac{\partial y_1}{\partial x_n} = 0$$

$$\frac{\partial y_2}{\partial x_1} = \cos x_1 \, \cos x_2 \quad \frac{\partial y_2}{\partial x_2} = -\sin x_1 \sin x_2 = \frac{\partial y_2}{\partial x_n} = 0$$

$$\frac{\partial y_n}{\partial x_1} = \cos x_1 \sin x_2 \ldots \cos x_n \ldots$$

$$\frac{\partial y_n}{\partial x_n} = -\sin x_1 \sin x_2 \ldots \sin x_n$$

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & 0 & 0 \ldots 0 \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & 0 \ldots 0 \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} \ldots \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \ldots \frac{\partial y_n}{\partial x_n}$$

$$= (-\sin x_1)(-\sin x_1 \sin x_2) \ldots (-\sin x_1 \sin x_2 \ldots \sin x_n)$$

$$= (-1)^n (\sin x_1)^n (\sin x_2^{n-1} \ldots \sin x_n)$$

H.W

4). If $x = U^2 - V^2$ $y = 2UV$ find $\frac{\partial(x,y)}{\partial(U,V)}$.

Solution :-

$x = U^2 - V^2 \quad y = 2UV$

$$\frac{\partial x}{\partial U} = 2U \qquad \frac{\partial y}{\partial U} = 2V$$

$$\frac{\partial x}{\partial V} = -2V \qquad \frac{\partial y}{\partial V} = 2U$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix}$$

$$= (4u^2 + 4v^2)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = 4(u^2+v^2)$$

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5). If $u+v=x$, and $u-v=y$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Solution:-

$u+v=x \qquad u-v=y$

$u=x-v \qquad v=u-y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \qquad \frac{\partial u}{\partial x}=1 \quad \frac{\partial u}{\partial y}=0 \qquad \frac{\partial v}{\partial x}=0 \quad \frac{\partial v}{\partial y}=-1$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1+0$$

$$\frac{\partial(u,v)}{\partial(x,y)} = -1$$

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6. If $u=x+y+z$

$v=xy+yz+zx$

$w=x^3+y^3+z^3-2xy$ then Prove that

$J(u,v,w)=0$.

Solution:

$u=x+y+z \quad v=xy+yz+zx$

$w=x^3+y^3+z^3-2xy$

$$J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 1 \quad \frac{\partial u}{\partial z} = 1$

$\frac{\partial v}{\partial x} = y+z \quad \frac{\partial v}{\partial y} = x+z \quad \frac{\partial v}{\partial z} = x+y$

$\frac{\partial w}{\partial x} = 3x^2 - 3yz \quad \frac{\partial w}{\partial y} = 3y^2 - 3xz \quad \frac{\partial w}{\partial z} = 3z^2 - 3xy$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix} \quad R_3 \to \frac{R_3}{3}$$

$$= 1\left[(x+z)(z^2-xy)-(x+y)(y^2-xz)\right]$$
$$-1\left[(y+z)(z^2-xy)-(x+y)(x^2-yz)\right]$$
$$+1\left[(y+z)(y^2-xz)-(x+y)(x^2-yz)\right] = 0$$

$= 1[(xz^2 - x^2y + z^3 - xyz) - (xy^2 - x^2z + y^3 - xyz)]$
$-1[(yz^2 - xy^2 + z^3 - xyz) - (x^3 - xyz + x^2y - y^2z)]$
$+1[(y^3 - xyz + y^2z - xz^2) - (x^3 - xyz + zx^2 - yz^2)]$

$= xz^2 - x^2y + z^3 - xyz - xy^2 - x^2z + y^3 + xyz$
$- yz^2 + xy^2 - z^3 + xyz - x^3 + xyz - x^2y + y^2z$
$+ y^3 - xyz + y^2z - xz^2 - x^3 + xyz - zx^2 + yz^2$

$= 0$

$J(u,v,w) = 0$ is Proved.

UNIT-2 Maxima and minima function of two variables.

The first and second derivative of a function of one variable can be used to derivative its maxima and minima.

III^ly The first order and second order partial can be used to determent maxima and minima function of two variables.

தேவையான நிபந்தனை :-
Necessary condition (maxima and minima):

$f(x,y)$ at $x = a$ and $y = b$ are

$f_x(a,b) = 0$ and $f_y(a,b) = 0$

where $f_x(a,b)$ and $f_y(a,b)$ respectives denote the values of $\left(\frac{\partial f}{\partial x}\right)$ and $\left(\frac{\partial f}{\partial y}\right)$ at $(x = a$ and $y = b)$

போதுமான நிபந்தனை :-
Sufficient conditions (Maxima and minima)

$f_x(a,b) = 0$ and $f_y(a,b) = 0$

& $r = f_{xx}(a,b)$
$s = f_{xy}(a,b)$
$t = f_{yy}(a,b)$.

i) If $rt-s^2>0$ and $r>0$, $f(x,y)$ is Minimum at (a,b).

ii) If $rt=s^2>0$ and $r<0$, $f(x,y)$ is maximum at (a,b)

iii) If $rt=s^2<0$ and $r>0$, $f(x,y)$ is (Saddle) neither maximum nor minimum.

iv) If $rt-s^2=0$ the case is doubtful.

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1). Find the maximum and minimum value of

$f(x,y) = 2(x^2-y^2) - x^4+y^4$ $\quad 2(1-0)-1+0$

$x^n = nx^{n-1}$ $\quad$ Solution:- $=(2x^2-2y^2)-x^4+y^4$. $\quad 2-1=1$

$\frac{\partial f}{\partial x} = 4x-4x^3 \qquad \frac{\partial f}{\partial y} = -4y+4y^3$

$r= \frac{\partial^2 f}{\partial x^2} = f_{xx} = 4-12x^2$

$s = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = 0$

$t = \frac{\partial^2 f}{\partial y^2} = f_{yy} = -4+12y^2$

For maximum and minimum value

$f_x=0 \quad f_y=0$

$4x(1-x^2)=0 \qquad 4y(y^2-1)=0$

$4x = \frac{0}{1-x^2}$ | $1-x^2=\frac{0}{4x}$ | $y^2-1=0$ | $4y=0$

$4x=0$ | $1-x^2=0$ | $y^2=1$ | $\boxed{y=0}$

$\boxed{x=0}$ | $x^2=1$ | $\boxed{y=\pm 1}$

$\boxed{x=\pm 1}$

$x=0 \qquad x=\pm 1$

$y=0 \qquad y=\pm 1$

$(0,0), (0,1), (0,-1), (1,0), (1,1), (1,-1), (-1,0), (-1,1)$
$(-1,-1)$

$rt - s^2 = (4-12x^2)(-4+12y^2) - 0$

$rt - s^2 = -16 + 48y^2 + 48x^2 - 144x^2y^2 \rightarrow (7)$.

$r = 4 - 12x^2$ $\quad x = 0 \quad y = 0$

$s = 0$

$t = -4 + 12y^2$ $\quad x = \pm 1 \quad y = \pm 1$

The points are:-

$(0,0), (0,1), (0,-1), (1,0), (1,1), (1,-1),$
$(-1,0), (-1,1), (-1,-1),$

* At the point $(\overset{x}{0}, \overset{y}{0})$

$r = 4 - 12(0)^2 = 4 - 0 = 4 > 0$

$s = 0$

$t = -4 + 12(0)^2 = -4 \Rightarrow$

$rt - s^2 = -16 < 0$

$\therefore (0,0)$ is a saddle point.

* At the point $(0,1)$

$r = 4 - 12(0)^2 = 4 - 0 = 4 > 0$

$s = 0$

$t = -4 + 12(1)^2 = -4 + 12 = 8 \Rightarrow rt - s^2 = (4 \times 8) - 0 = 32 > 0$.

$\therefore (0,1)$ is a minimom point.

* At the point $(0,-1)$.

$r = 4 - 0 = 4 > 0$

$s = 0$

$t = -4 + 12(-1)^2 = -4 + 12 = 8$

$rt - s^2 = (4 \times 8) - (0)^2 = 32 > 0$

$\therefore (0,-1)$ is a minimom point.

* At the point $(-1,1)$

$r = 4-12(-1)^2 = 4-12 = -8 < 0$

$s = 0$

$t = -4+12(1)^2 = -4+12 = 8 > 0$

$rt - s^2 = (-8 \times 8) - 0 = -64 < 0$

$\therefore (-1,1)$ is a saddle point.

* At the point $(-1,-1)$.

$r = 4-12(-1)^2 = 4-12 = -8 < 0$

$s = 0$

$t = -4+12(-1)^2 = -4+12 = 8 > 0$

$rt - s^2 = -64 < 0$

$\therefore (-1,-1)$ is a saddle point.

* At the point $(1,0)$.

$r = 4-12(1)^2 = 4-12 = -8 < 0$

$s = 0$

$t = -4+12(0) = -4 < 0$

$rt - s^2 = (-8 \times -4) - 0 = 32 > 0$

$\therefore (1,0)$ is a maximum point.

* At the point $(1,1)$.

$r = 4-12(1)^2 = 4-12 = -8 < 0$

$s = 0$

$t = -4+12(1)^2 = -4+12 = 8 > 0$

$rt - s^2 = (-8 \times 8) - 0 = -64 < 0$

$\therefore (1,1)$ is a saddle point.

* At the point $(1,-1)$

$r = 4 - 12(1)^2 = 4 - 12 = -8 < 0$

$s = 0$

$t = -4 + 12(-1)^2 = -4 + 12 = 8 > 0$

$rt - s^2 = (-8 \times 8) - 0 = -64 < 0.$

$\therefore (1,-1)$ is a saddle point.

*. AT the point $(-1,0)$.

$r = 4 - 12(-1)^2 = 4 - 12 = -8 < 0$

$s = 0$

$t = -4 + 12(0) = -4 < 0$

$rt - s^2 = (-8 \times -4) - 0 = +32 > 0$

$\therefore (-1,0)$ is a maximum point.

Function is minimum at $(0, \pm 1)$

Minimum Value :-

$f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

$= 2x^2 - 2y^2 - x^4 + y^4$

$f(\overset{x}{0}, \overset{y}{\pm 1}) = 2(0)^2 - 2(1)^2 - (0)^4 + (1)^4$

$f(0,1) = -2 + 1 = -1$

$f(0,-1) = 2(0)^2 - 2(-1)^2 - (0)^4 + (-1)^4$

$= -2 + 1 = -1$

Function is maximum at $(\pm 1, 0)$

Maximum value.

$f(x,y) = 2x^2 - 2y^2 - x^4 + y^4$

$f(1,0) = 2(1)^2 - 2(0)^2 - (1)^4 + (0)^4$

$= 2 - 1 = 1$

$f(-1,0) = 2(-1)^2 - 2(0)^2 - (-1)^4 + (0)^4$

$= 2 - 1 = 1.$

Unit-2

2) Method of Lagranges type Find the maxima and minima if any of the function is.

$\Rightarrow f(x,y) = 12xy - 3y^2 - x^2$ subject $\boxed{x+y=16}$

$\Rightarrow g(x,y)$

Solution:

$$f(x,y) = 12xy - 3y^2 - x^2$$

$$g(x,y) = x+y-16$$

Concider $F(x,y,\lambda) = f(x,y) - \lambda(g(x,y)$

$$= 12xy - 3y^2 - x^2 - \lambda(x+y-16)$$

$$= 12xy - 3y^2 - x^2 - x\lambda - y\lambda + 16\lambda$$

$$\frac{\partial F}{\partial x} = 12y - 2x - \lambda$$

$$\frac{\partial F}{\partial y} = 12x - 6y - \lambda$$

$$\frac{\partial F}{\partial z} = -x - y + 16$$

Solve :- $\frac{\partial F}{\partial x} = 0 \quad \frac{\partial F}{\partial y} = 0 \quad \frac{\partial z}{\partial y} = 0$

$$\therefore -2x + 12y - \lambda = 0 \longrightarrow (1)$$

$$12x - 6y - \lambda = 0 \longrightarrow (2)$$

$$x + y - 16 = 0 \longrightarrow (3)$$

(2) - (1) $\Rightarrow$

$$12x - 6y - \lambda = 0$$
$$-2x + 12y - \lambda = 0$$
$$(+) \quad (-) \quad (+)$$

$$14x - 18y = 0$$

$$7x - 9y = 0$$

$③ \times 7 \Rightarrow \quad 7x+7y-112=0$

$\quad\quad\quad\quad 7x-9y \quad\quad = 0$

$\quad\quad\quad\quad (-) \quad (+)$

$16y = 112$

$y = \frac{\cancel{112}\ 7}{\cancel{16}}$

$\boxed{y=7}$

$\therefore y=7 \rightarrow ③.$

$x+7-16=0$

$x-9=0$

$\boxed{x=9}$

When $y=7$, $x=9$

$\therefore (9,7)$ is an extremum.

point $f(9,7) = 12(7\times9) - 3(7)^2 - (9)^2$

$f(9,7) = (12\times63) - (3\times49) - 81$

$= 756 - 147 - 81$

$= 528$

$(0,16)$ is a point satisfying the constraint.

$x+y=16 \Rightarrow x=0 \Rightarrow 0+y=16 \quad y=16$

$\cancel{f(0\times16) = 12(0\times16) - 3(16)^2 - (16)^2}$

$f(0,16) = 3\times256 = -768.$

$f(9,7) > f(0,16)$

$\therefore$ The function $f(x,y)$ is maximum at $(9,7)$ and the maximum value is 528

2) The rectangular box without a lid is to be made from 12 m3 of cardboard. Find the volume of such a box.

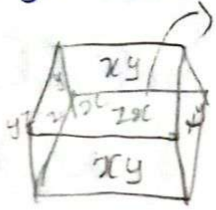
Solution:

Let x, y, z be the dimensions of the box

and $g(x,y,z) = 12 \quad V = x, y, z$

$F(x,y,z,\lambda) = V(x,y,z) - \lambda g(x,y,z)$

$F(x,y,z,\lambda) = xyz - \lambda(xy + 2yz + 2zx - 12)$



$$\frac{\partial F}{\partial x} = yz - y\lambda - 2z\lambda$$

$$\frac{\partial F}{\partial y} = xz - x\lambda - 2z\lambda$$

$$\frac{\partial F}{\partial z} = xy - 2y\lambda - 2x\lambda$$

$$\frac{\partial F}{\partial \lambda} = -(xy + 2yz + 2zx - 12)$$
$$= -xy - 2yz - 2zx + 12$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yz - y\lambda - 2z\lambda = 0$$
$$\boxed{yz = \lambda(y + 2z)} \longrightarrow (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow xz - 2y\lambda - 2z\lambda = 0$$
$$\boxed{zx = \lambda(x + 2z)} \longrightarrow (2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow xy - 2y\lambda - 2x\lambda = 0$$
$$\boxed{xy = \lambda(2y + 2x)} \longrightarrow (3)$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow -xy - 2yz - 2zx + 12 = 0$$
$$\boxed{xy + 2yz + 2zx = 12} \longrightarrow (4)$$

Multiply eqw ① x

$xyz = \lambda(xy + 2xz) \longrightarrow$ ⑤

Multiply eqw ② y

$xyz = \lambda(xy + 2zy) \longrightarrow$ ⑥

Multiply eqw ③ y

$xyz = \lambda(2xz + 2yz) \longrightarrow$ ⑦

$\therefore$ when $\lambda = 0 \Rightarrow x, y, z = 0$

From ⑤ & ⑥ when $\lambda \neq 0$

$\cancel{xy} + 2x\cancel{z} = \cancel{xy} + 2\cancel{z}y \quad z \neq 0$

$\boxed{x = y}$

From ⑥ and ⑦ when $\lambda \neq 0$

$2\cancel{x}z + 2yz = \cancel{x}y + 2zy$

$\boxed{2z = y}$

$\therefore x = y = 2z$

From ④

$(2z)(2z) + 2(2z)z + 2z(2z) = 12$

$4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow z = 1$

$12z^2 = 12$

$z = 1, x = 2, y = 2$

$\therefore$ Volume $= xyz = (2)(2)(1)$

$\boxed{V = 4\,m^3}$

2. Find the shortest distance from the pt $(1,0,-2)$ to the plane $x+2y+z=4$

Solution:

distance formula

$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

distance from the pt (x,y,z) to the point $(1,0,-2)$ is

$$d = \sqrt{(x-1)^2+(y-0)^2+(z+2)^2}$$

$$d^2 = (x-1)^2+(y)^2+(z+2)^2 \longrightarrow ①$$

$$z = 4-x-2y \longrightarrow ②$$

Sub ② in ①.

$$f(x,y) = d^2 = (x-1)^2+y^2+(6-x-2y)^2$$

$$= x^2+1-2x+y^2+(6-x-2y)^2$$

$$f_x = 2(x-1)-2(6-x-2y)$$

$$= 2x-2-12+2x+4y$$

$$= 4x+4y-14$$

$$f_y = 2y-4(6-x-2y)$$

$$= 2y-24+4x+8y$$

$$= 4x+10y-24$$

$$f_{xx} = 4$$

$$f_{xy} = -4$$

$$f_{yy} = 10$$

$$f_x = 0 \qquad f_y = 0$$

$4x+4y=14 \rightarrow ①$ $\quad 4x+10y=24 \rightarrow ②$

$① \Rightarrow 4x+4y=14$

$② \Rightarrow 4x+10y=24$

$\quad (-) \quad (-) \quad (-)$

$-6y=-10$

$\boxed{y=10/6} \Rightarrow y=5/3$

Sub in ① $\Rightarrow 4x+4y=14 \quad = 4x+4y=14$

$4x+4(5/3)=14$

$4x=14-20/3=\frac{42-20}{3}=\frac{22}{3}$

$4x=22/3$

$x=\frac{22}{3\times4}$

$x=\frac{22}{12}$

$\boxed{x=11/6}$

$\therefore (11/6, 5/3)$

$\therefore$ The critical point is $(11/6, 5/3)$

$\Rightarrow z \Rightarrow$

$\boxed{z=4-x-2y}$

$z=4-(11/6)-2(5/3)$

$z=\frac{24-11}{6}-\frac{10}{3}$

$z=\frac{13-20}{6}$

$z=\frac{-7}{6}$

$z=\frac{-7}{6}$

$\boxed{z=5/6}$

$d=\sqrt{(5/6)^2+(5/3)^2+(5/6)^2}$ $\quad d=\sqrt{\frac{25+100+25}{6}}$

$=\frac{5\sqrt{6}}{6}$ units. $\quad =\frac{\sqrt{150}}{6}=\frac{\sqrt{25\times6}}{6}$

$d=\frac{5\sqrt{6}}{6}$

$11/6 - 1$

$\frac{11-6}{6}$

$5/9$

$150=25\times6$

$\frac{5\sqrt{6}}{6}$

---

4. Investigate the maximum and minimum value of $4x^2+6xy+9y^2-8x-24y+4$

Solution:

Given that $f(x,y)$

$f(x,y)=4x^2+6xy+9y^2-8x-24y+4$

$f_x = 8x+6y-8$

$f_y = 6x+18y-24$

$f_{xx} = r = 8 \Rightarrow \frac{\partial^2 f}{\partial x^2}$

$f_{yy} = t = 18 \Rightarrow \frac{\partial^2 f}{\partial y^2}$

$f_{xy} = s = 6 \Rightarrow \frac{\partial^2 f}{\partial x \partial y}$

For maximum and mininum value.

$f_x = 0 \qquad f_y = 0$

$8x+6y-8=0 \rightarrow (1) \qquad 6x+18y-24=0 \rightarrow (2)$

$(1)\times 3 \Rightarrow 24x + \cancel{18y} - \cancel{24} = 0$

$(2) \Rightarrow \quad 6x + \cancel{18y} - \cancel{24} = 0$

$\quad (-) \quad (-) \quad (+)$

$18x = 0$

$\boxed{x=0}$

Sub $x=0$ in (1).

$8(0)+6y-8=0$

$6y-8=0$

$6y=8$

$y=8/6$

$\boxed{y=4/3}$

$\therefore (0, 4/3)$

$rt-s^2 = (8)(18)-36$

$= 144-36$

$= 108 > 0$

$r > 0$

$\therefore$ Function is minimum at $(0, 4/3)$

minimum value.

$f(x,y) = 4x^2+6xy+9y^2-8x-24y+4$

$f(0, 4/3) = 0+0+9(16/9)-0-24(4/3)+4$

$= 16+4-32 = -12$

$\therefore$ minimum value $= -12$

5). Find the minimum value of $x^2+5y^2-6x+10y+12$

Solution:

$$f(x,y) = x^2+5y^2-6x+10y+12$$

$$\frac{\partial f}{\partial x} = f_x = 2x-6$$

$$\frac{\partial f}{\partial y} = f_y = 10y+10$$

$$r = \frac{\partial^2 f}{\partial x^2} = f_{xx} = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = f_{yy} = 10$$

For minimum value,

$f_x = 0$ $\qquad$ $f_y = 0$

$2x-6=0$ $\qquad$ $10y+10=0$

$2x=6$ $\qquad$ $\boxed{y=-1}$

$\boxed{x=3}$

$\therefore (3,-1)$

$$rt-s^2 = (2)(10)-0$$
$$= 20>0;$$

$r>0$

$\therefore$ Function is minimum at $(3,-1)$

$$f(x,y) = x^2+5y^2-6x+10y+12$$

$$f(3,-1) = 9+5y^2-6x+10y+12$$
$$= 26-28 = -2$$

$\therefore$ minimum value of -2//

Method of Lagrange's multipliers

1). Find the maxima and minima of the function of $f(x,y) = 3x^2+4y^2-xy$, if $2x+y=21$

Solution:

$$f(x,y) = 3x^2+4y^2-xy$$
$$g(x,y) = 2x+y-21=0$$

$$F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$
$$= 3x^2+4y^2-xy-\lambda(2x+y-21)$$

$$\frac{\partial F}{\partial x} = 6x-y-2\lambda$$

$$\frac{\partial F}{\partial y} = 8y-x-\lambda$$

$$\frac{\partial F}{\partial \lambda} = -(2x+y-21)$$
$$= -2x-y+21$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 6x-y-2\lambda=0 \longrightarrow (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 8y-x-\lambda=0 \longrightarrow (2)$$

or

$x-8y+\lambda=0$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow -2x-y+21=0 \longrightarrow (3)$$

Solve (1) and (2)

$$(1) \Rightarrow 6x-y-\cancel{2\lambda}=0$$
$$(2)\times 2 \quad -2x-16y-\cancel{2\lambda}=0$$
$$(+)\ (-)\ (+)$$
$$8x-17y=0 \longrightarrow (4)$$

Solve (3) and (4)

$$3\times 4 \Rightarrow -\cancel{8x}-4y+84=0$$
$$\cancel{8x}-17y=0$$
$$-21y+84=0$$

$-21y = -84$

$y = \frac{\cancel{84}^{\cancel{28}\,4}}{\cancel{21}_{\cancel{7}\,1}}$ $\boxed{y = 4}$

Sub $y = 4$ in (4).

$8x - 17 \times (4) = 0$

$8x - 68 = 0$

$8x = 68$

$x = \frac{\cancel{68}^{17}}{\cancel{8}_{2}}$

$\boxed{x = 17/2}$

$\therefore (17/2, 4)$

$f(17/2, 4) = 3(17/2)^2 + 4(4)^2 - (17/2)(4)^2$

$= \frac{3 \times 289}{4} + 4 \times 16 - 34$

$= \frac{867}{4} + 64 - 34$

$= \frac{867}{4} + 30$

$= \frac{867 + 120}{4} = \frac{987}{4}$

17×17, 119, 17, 289, 289×3, 867

64×4, 256

34×4, 136

$(0, 21)$ is a pt satisfing constant

$2x + y = 21$

$f(0, 21) = 3(0) + 4(441) - 0$

$= 1764 > \frac{987}{4}$

$\therefore F(x, y)$ is a minimum at

$(17/2, 4)$ is $\frac{987}{4}$,,

11. Find the extreme value of the function $f(x,y) = x^2+y$ on the circle $x^2+y^2=1$.

Solution:

$$f(x,y) = x^2+y$$
$$g(x,y) = x^2+y^2-1=0$$

$$F(x,y,\lambda) = f(x,y) - \lambda g(x,y).$$
$$= x^2+y-\lambda(x^2+y^2-1)$$
$$= x^2+y-\lambda x^2 - \lambda y^2 + \lambda$$

$$\frac{\partial F}{\partial x} = 2x - 2\lambda x$$

$$\frac{\partial F}{\partial y} = 1-2\lambda y$$

$x = 2\lambda x$

$2x = 2x(\lambda)$

$$\frac{\partial F}{\partial \lambda} = -(x^2+y^2+1)$$
$$= -(x^2+y^2-1)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x - 2\lambda x = 0$$
$$2x = 2\lambda x \longrightarrow (1).$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 1 = 2\lambda y \longrightarrow (2)$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x^2+y^2 = 1 \longrightarrow (3).$$

From (1) $x=0$ or $\lambda = 1$.

If $x=0$ from (3) we get

$y = \pm 1$

If $\lambda = 1$ from (2) we get

$y = 1/2$

From (3) we get

$x = \pm \sqrt{3}/2$

$x^2 + 1/4 = 1$

$x^2 = 1 - 1/4$

$x^2 = \frac{4-1}{4}$

$x^2 = 3/4$

$x = \pm\sqrt{3}/2$

∴ The extreme points are
$(0,1), (0,-1), (\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, 1/2)$.

we note that $f(0,1) = 1$

$f(0,-1) = -1$

$f(\sqrt{3}/2, 1/2) = 5/4$

$f(-\frac{\sqrt{3}}{2}, 1/2) = 5/4$

$f(x,y) = x^2+y$
$f(0,1) = (0)^2+1 = 4$

The maximum value of F on the circle $x^2+y^2=1$ is $5/4$ and the minimum value is -1.

---

2) Find the points on the sphere $x^2+y^2+z^2=4$ that are closest and farthest from the point $(3,1,-1)$.

Solution:

The distance between the points (x,y,z) and $(3,1,-1)$ is

$$d = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$$

$$d = \sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$$

ie $d^2 = (x-3)^2+(y-1)^2+(z+1)^2 = f(x,y,z)$

$g(x,y,z) = x^2+y^2+z^2-4 = 0$.

consider the function

$$F(x,y,z;\lambda) = f(x,y,z) - \lambda g(x,y,z)$$

$$= (x-3)^2+(y-1)^2+(z+1)^2 - \lambda(x^2+y^2+z^2-4)$$

$$= (x-3)^2+(y-1)^2+(z+1)^2 - \lambda x^2 - \lambda y^2 - \lambda z^2 + 4$$

$$\frac{dF}{dx} = 2(x-3) - 2\lambda x$$

$\frac{\partial F}{\partial y} = 2(y-1) - 2\lambda y$

$\frac{\partial F}{\partial z} = 2(z+1) - 2\lambda z$

$\frac{\partial F}{\partial \lambda} = \cancel{\lambda}(-x^2 + y^2 - z^2 + 4$

$= -(x^2 + y^2 + z^2 - 4)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2(x-3) - 2\lambda x = 0$

$2x - 6 = \lambda x$ $\quad 2(x-3) = 2\lambda x \longrightarrow (1)$

$[x - 3 = \lambda x]$ $\quad x = 3$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2(y-1) - 2\lambda y = 0$

$2(y-1) = 2\lambda y \longrightarrow (2)$.

$[(y-1) = \lambda y]$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2(z+1) - 2\lambda z = 0$

$2(z+1) = 2\lambda z \longrightarrow (3)$.

$(z+1) = \lambda z$

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x^2 + y^2 + z^2 = 4 \longrightarrow (4)$

Solving for λ from (1), (2), (3) we get.

$x = \frac{3}{1-\lambda} \quad y = \frac{1}{1-\lambda} \quad z = \frac{1}{1-\lambda}$

Sub in (4) we get

$$\frac{9}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 4$$

$$\frac{11}{(1-\lambda)^2} = 4$$

ie $4(1-\lambda)^2 = 11$ or $\lambda = 1 \pm \frac{\sqrt{11}}{2}$

when $\lambda = 1 + \sqrt{11}/2$ the point (x, y, z) is

$\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

Smallest value at the points $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$

$\therefore$ Closest points is $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$

farthest points is $\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

Q. using lagrang's multipliers method find maximum and minimum value of $F(x, y) = x^2 - y$ subject to $x^2 + y^2 = 1$.

Solution:

$$F(x, y) = x^2 - y^2$$

$$g(x, y) = x^2 + y^2 - 1 = 0$$

$$F(x, y, \lambda) = x^2 - y^2 - \lambda(x^2 + y^2 - 1)$$

$$= x^2 - y^2 - \lambda x^2 - \lambda y^2 + \lambda$$

$$\frac{\partial F}{\partial x} = 2x - 2\lambda x$$

$$\frac{\partial F}{\partial y} = -2y - 2\lambda y$$

$$\frac{\partial F}{\partial \lambda} = -x^2 - y^2 + 1$$

$$= -(x^2 + y^2 - 1)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x - 2\lambda x = 0$$

$$2x = 2\lambda x \longrightarrow (1).$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y - 2\lambda y = 0.$$

$$2y = 2\lambda y \longrightarrow (2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow -(x^2+y^2-1) = 0$$

$$x^2+y^2 = 1 \longrightarrow (3)$$

$2x - 2\lambda x \Rightarrow (1)\ x = 0$ or $\lambda = 1$

$2y - 2\lambda y \Rightarrow -y = 0$ or $\lambda = -1$

when $x = 0,\ y = \pm 1$

$y = 0,\ x = \pm 1$

The extreme points $(0,1), (0,-1), (-1,0)$
$(1,0)$

$f(0,1) = -1$

$f(0,-1) = -1$

$f(1,0) = 1$

$f(-1,0) = 1$

(+) $\therefore$ Maximum value $= F(\pm 1, 0) = 1$

(-) Minimum value $= F(0, \pm 1) = -1$,

---

4. For the shortest and the longest distance from the pts $(1,2,-1)$ to the sphere $x^2+y^2+z^2 = 24$.

Solution:

$$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z+1)^2$$

$$g(x,y,z) = x^2+y^2+z^2-24 = 0$$

$$F(x,y,z,\lambda) = (x-1)^2 + (y-2)^2 + (z+1)^2 - \lambda(x^2+y^2+z^2-24)$$

$$= (x-1)^2 + (y-2)^2 + (z+1)^2 - \lambda x^2 - \lambda y^2 - \lambda z^2 + \lambda 24$$

$$\frac{\partial F}{\partial x} = 2(x-1) - 2\lambda x$$

$$\frac{\partial F}{\partial y} = 2(y-2) - 2\lambda y$$

$$\frac{\partial F}{\partial z} = 2(z+1) - 2\lambda z$$

$$\frac{\partial F}{\partial \lambda} = -\cancel{x}^2 - y^2 - z^2 + 24$$

$$= -(x^2+y^2+z^2-24)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2(x-1) = 2\lambda x$$

$$\therefore x - 1 = \lambda x \Rightarrow x - \lambda x = 1$$

$$x(1-\lambda) = 1$$

$$x = \frac{1}{1-\lambda} \rightarrow ①$$

$$x = \frac{1}{1-\lambda} \rightarrow ① ; \; y = \frac{2}{1-\lambda} \rightarrow ② \; z = \frac{-1}{1-\lambda} \rightarrow ③$$

$$x^2+y^2+z^2 = 24$$

$$\left(\frac{1}{(1-\lambda)}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right) = 24$$

$$\frac{1+4+1}{(1-\lambda)^2} = 24$$

$$\frac{6}{(1-\lambda)^2} = 24$$

$$\frac{6}{24} = (1-\lambda)^2$$

$$(1-\lambda)^2 = \frac{1}{4}$$

$$x^2+y^2+z^2 = 24$$

$$\therefore (1-\lambda)^2 = 1/4$$

$$1 - \lambda = \pm 1/2$$

$$\lambda = 1 \pm 1/2 = 3/2 \text{ (or) } 1/2$$

When $\lambda = 1/2$ the points is $(2, 4, -2)$

When $\lambda = 3/2$ the points is $(-2, -4, 2)$

$f(2,4,-2) = 1+4+1 = 6$ $\quad d^2 = f(x,y,z)$

$d^2 = 6 \Rightarrow d = \sqrt{6}$

$f(-2,-4,2) = 9+36+9 = 54$ $\quad d^2 = 54 \Rightarrow d = \sqrt{54}$.

Minimum distance $= \sqrt{6}$

Maximum distance $= \sqrt{54} = \sqrt{9 \times 6} = 3\sqrt{6}$

(*) Example 2. If $x = r\cos\theta$, $y = r\sin\theta$, then find the Jacobian of x and y with respect to r and θ.

(or)

In polar coordinates, $\frac{\partial(x,y)}{\partial(r,\theta)}$

Solution:

The Jacobian of x and y with respect to r and θ is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Given $x = r\cos\theta$

$$\therefore \frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

and $y = r\sin\theta$.

$$\therefore \frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r(1)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$ //

(H.W) Examine $f(x,y) = x^3+y^3-12x-3y+20$ for its extreme values

(minimum And maximum)

Solution:

Given $f(x,y) = x^3+y^3-12x-3y+20$

$$\frac{\partial f}{\partial x} = 3x^2-12 \qquad \frac{\partial f}{\partial y} = 3y^2-3$$

$$r = \frac{\partial^2 f}{\partial x^2} = f_{xx} = 6x$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = f_{yy} = 6y$$

For maximum and minimum value: [extreme values].

$f_x = 0 \qquad f_y = 0$

$3x^2-12=0 \qquad\qquad 3y^2-3=0$

$\cancel{3x^2=12}$

$3(x^2-4)=0 \qquad\qquad 3(y^2-1)=0$

$x^2-4=0/3 \qquad\qquad y^2-1=0/3$

$x^2-4=0 \qquad\qquad y^2-1=0$

$x^2=4 \qquad\qquad y^2=1$

$x=\sqrt{4} \qquad\qquad y=\sqrt{1}$

$\boxed{x=\pm 2} \qquad\qquad \boxed{y=\pm 1}$

$x=2 \qquad x=-2$

$y=1 \qquad y=-1$

$(2,1), (2,-1), (-2,1), (-2,-1)$

$rt - s^2 = (6x)(6y) - 0$

$= 36xy \longrightarrow (1)$

$r = 6x$ $\quad x = +2 \quad y = 1$

$s = 0$

$t = 6y$ $\quad x = -2 \quad y = -1$

The points are:

$(2,1), (2,-1), (-2,1), (-2,-1).$

*. At the point $(2,1)$

$r = 6(2) = 12 > 0$

$s = 0$

$t = 6(1) = 6 > 0$

$rt - s^2 = (12)(6) - 0$

$rt - s^2 = 72 > 0$

$\therefore (2,1)$ is a minimum point.

*. At the poin $(2,-1)$

$r = 6(2) = 12 > 0$

$s = 0$

$t = 6(-1) = -6 < 0$

$rt - s^2 = (12)(-6) - 0$

$rt - s^2 = -72 < 0$

$(2,-1)$ is a saddle point.

At the point $(-2,+1)$

$r = 6(-2) = -12 < 0$

$s = 0$

$t = 6(1) = 6 > 0$

$rt - s^2 = (-12)(6) - 0$

$rt - s^2 = -72 < 0$

$\therefore (-2,1)$ is a Saddle point.

At the point $(-2,-1)$.

$r = 6(-2) = -12 < 0$

$s = 0$

$t = 6(-1) = -6 < 0$

$rt - s^2 = (-12)(-6) - 0$

$rt - s^2 = 72 > 0$

$(-2,-1)$ is a maximum point.

Function is minimum at $(2,1)$.

minimum value:

$f(x,y) = x^3 + y^3 - 12x - 3y + 20$

$f(2,1) = 2^3 + 1^3 - 12(2) - 3(1) + 20$

$= 8 + 1 - 24 - 3 + 20$

$= 29 - 27$

$f(2,1) = 2$ //

Function is maximum at $(-2,-1)$

maximum value:

$f(-2,-1) = (-2)^3 + (-1)^3 - 12(-2) - 3(-1) + 20$

$f(-2,-1) = -8 - 1 + 24 + 3 + 20$

$= 47 - 9$

$f(-2,-1) = 38$ //

V.IP.

Example 3

Find the maximum and minimum values of $f(x,y) = x^4 + y^4 - 4xy + 1$

Solution:

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$f_x = \frac{\partial f}{\partial x} = 4x^3 - 4y \; ; \left[\frac{\partial^2 f}{\partial x^2} = 12x^2 = r\right] = f_{xx}$$

$$f_y = \frac{\partial f}{\partial y} = 4y^3 - 4x \; ; \left[\frac{\partial^2 f}{\partial y^2} = 12y^2 = t\right] = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4 \; [s] = f_{xy}$$

For maximum and minimum value.

$$f_x = 0 \quad f_y = 0$$

$4x^3 - 4y = 0$	$4y^3 - 4x = 0$
$4(x^3 - y) = 0$	$4(y^3 - x) = 0$
$x^3 - y = 0$	$y^3 - x = 0$

$$x^9 - x = 0 \text{ or } x(x^8 - 1) = 0$$

$$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$

The real roots of x are $x = 0, 1, -1$.

The real values of y are $y = 0, 1, -1$.

$\therefore$ the critical points are $(0,0)$, $(1,1)$, $(-1,-1)$.

At $(0,0)$, $rt-s^2 = -16 < 0$

At $(1,1)$, $rt-s^2 = 12 \times 12 - 16 > 0$

At $(-1,-1)$, $rt-s^2 = 144 - 16 > 0$

Since $rt-s^2 < 0$ at $(0,0)$, it is a Saddle point, i.e. f is neither maximum nor minimum at $(0,0)$.

At $(1,1)$ and $(-1,-1)$, $rt-s^2 > 0$ and $r > 0$.

$\therefore$ the function is minimum at $(1,1)$ and $(-1,-1)$

Minimum value at $(1,1)$ is $f(1,1) = -1$

Minimum value at $(-1,-1)$ is $f(-1,-1) = -1$