19UMA05

STATICS

B.Sc. MATHEMATICS

III - SEMESTER

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UNIT-$\hat{\text{II}}$

COUPLES,

MOMENTS OF A COUPLE,

THEOREMS ON COUPLES,

PROBLEMS.

[Chapter IV (section 1 to 10)].

UNIT - II

COUPLES

Defenition :- COUPLES

Two equal and unlike parallel forces not acting at the same point are said to constitute a couple.

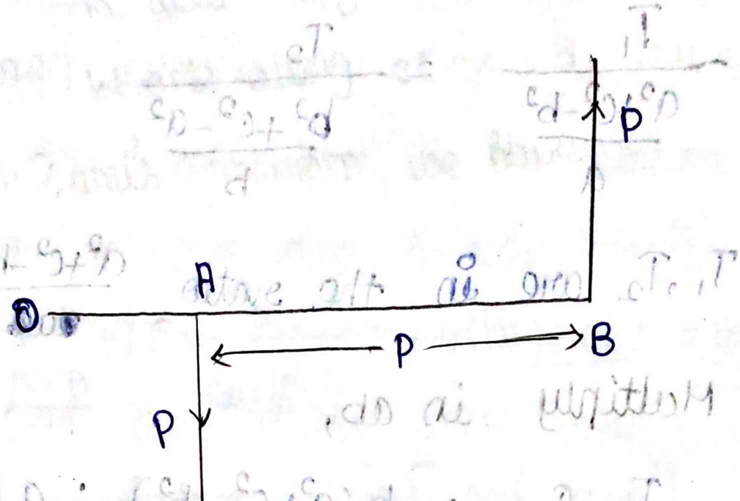
Example :-

(1) Winding a clock

(2) Turning a tap.

DEF :- MOMENT OF A COUPLE

The moment of a couple is the product of either of the two forces of the couple and the perpendicular distance between them.



DEF : Arm of a couple :-

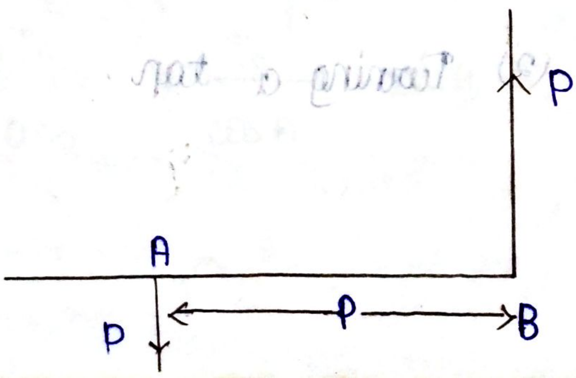
The perpendicular distance AB (= P) between the two equal forces P of a couple is called the arm of the couple.

A couple each of whose forces is P and whose arm is P as to figure is denoted by (P, P).

BOOK WORK

Show that the moment of the couple is independent of the point about which the moment is obtained.

Proof :-



Let P, P be the magnitudes of the forces forming a couple and O be any point in their plane.

Draw OAB Perpendicular to the forces to meet their lines of action in A and B.

The algebraic sum of the moments of the forces about O is

$= P.OB - P.OA \quad = P(OB - OA)$

$= P.(AB)$

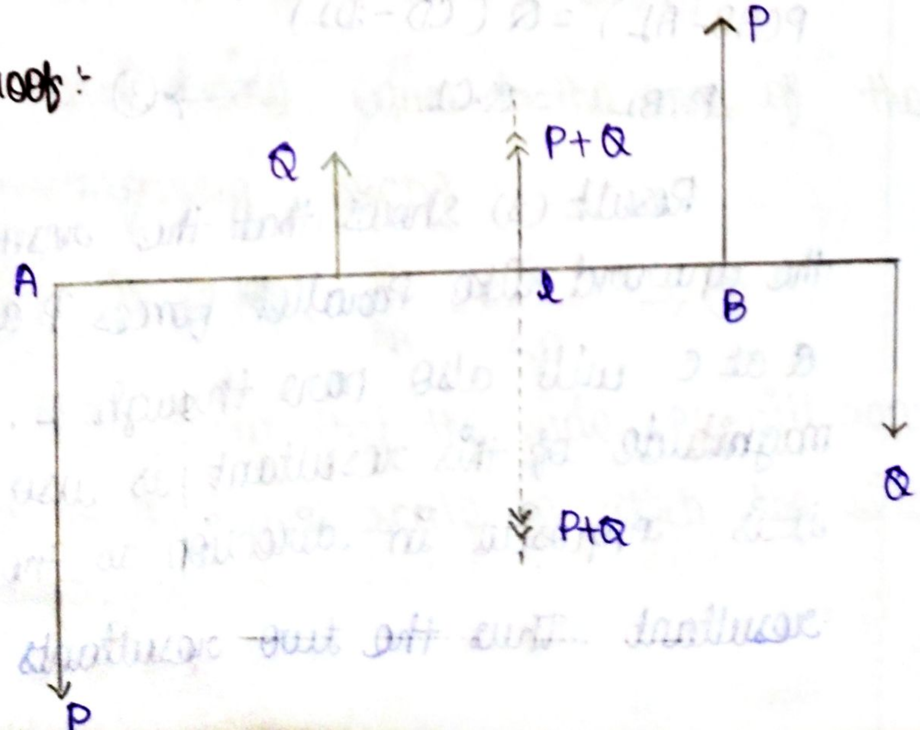
and this value is independent of the Position of O.

Equilibrium of Two couples

Theorem:

If two couples, whose moments are equal and opposite, act in the same plane upon a rigid body, the balance one another.

Proof:-



Let (P,P) and (Q,q) be two given couples such that $Pp = Qq$ in magnitude but opposite in sign.

Case(i) :-

Let the forces P and Q be Parallel

Draw a straight line Perpendicular to the lines of action of the forces, meeting them at A, B, C, D as in figure.

Since the moments of the couples are equal. we have,

$$P \cdot AB = Q \cdot CD \rightarrow (1)$$

The downward like parallel forces P at A and Q at D can be compounded into a single forces P+Q acting at L such that

$$P \cdot AL = Q \cdot DL \rightarrow (2)$$

(1)-(2) gives

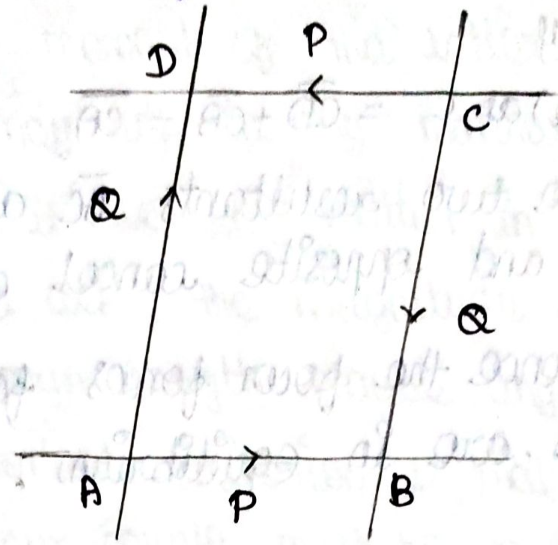
$$P(AB - AL) = Q(CD - DL)$$

$$P \cdot BL = Q \cdot CL \rightarrow (3)$$

Result (3) shows that the resultant of the upward like Parallel forces P at B and Q at C will also pass though L. The magnitude of this resultant is also (P+Q) but it is opposite in direction to the previous resultant. Thus the two resultants balance each other. Hence the four forces forming the couples are in equilibrium.

case (2) :-

Let the forces P and Q intersect



Let the two forces P of the couple (P,P) meets the two forces Q of the couple (Q,q) at the point A, B, C, D clearly ABCD is a parallelogram.

Let AB represent P on same scale AS the moments of the two couples are equal.

We have $Pp = Qq$ → ①

Also $AB \cdot P = AD \cdot q$ → ②

[each being equal to the area of the Parallelogram ABCD]

① - ② gives, $\frac{P}{AB} = \frac{Q}{AD}$ → ③

③ show that the side AD will represent Q on the same scale in which the side AB represent P.

The two forces P and Q meeting at A can be compounded by Parallelogram law so that

$$(P+Q) \text{ at } A = \overline{AB} + \overline{AD} = \overline{AC}$$

similarly

$$(P+Q) \text{ at } C = \overline{CD} + \overline{CB} = \overline{CA}$$

The two resultants $\overline{AC}$ and $\overline{CA}$ being equal and opposite cancel each other.

Hence the four forces forming the couples are in equilibrium.

Equivalence of two couples

Theorem:-

Two couples in the same plane whose moment are equal and of the same sign are equivalent to one another.

Let (P,P) and (Q,q) between couples in one plane having the same equal moments in magnitude and direction. Let (R,r) be a third couple in the same plane, whose moments is equal to the moment of either (P,P) or (Q,q) only in magnitude but opposite in direction. By the previous theorem, the couple (R,r) will balance the couple (P,P). It will also balance the couple (Q,q). Hence the effects of the couples (P,P) and (Q,q) must be the same. In other words, they are equivalent.

This is a fundamental theorem on coplanar couples. From this, it follows that a couple in a plane canbe replaced by any other couple in the same plane, provided that the moment of the latter replacing couple may act in any manner in that plane. ie it does not matter in what direction its forces act; the magnitude of its forces and its arm length maybe any thing. The only important criterion is that the moment of the new couple must be equal to that of the first couple in magnitude and sense.

Thus, a couple (P, P) may be replaced by a couple $\left(F, \frac{Pp}{F}\right)$ in the same plane with its constituent forces each equal to F and the arm length being equal to $\frac{Pp}{F}$. The moment of this couple is $F \frac{Pp}{F} = P.p$ moment of the first couple. Also one force F may be taken to be acting in any line and directions the other at the distance $\frac{Pp}{F}$ being on that side so as to make the sign of the moment same as that of (P, P)

Similarly, the couple (P, P) may be replaced by a couple $\left(\frac{Pp}{x}, x\right)$ with a given arm x any where in the plane.

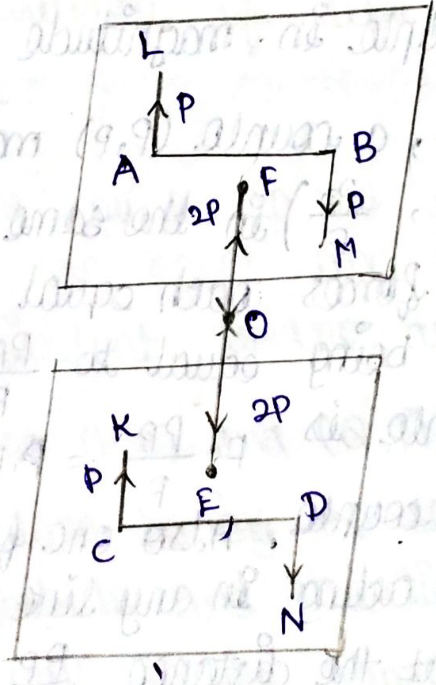
Couples in Parallel Planes

Theorem:-

The effect of a couple upon a rigid body is not altered if it is transfered to a Parallel Plane Provided its moment remain unchanged in magnitude and direction.

Proof:-

Consider a couple of forces P at the ends of a aim AB in a given plane.



Let AL and BM be the lines of action of the forces.

In any Parallel Plane, take a straight line CD equal and Parallel to AB.

Then ~~ABCD~~ ABDC will be a parallelogram the diagonals AD and BC will bisect each other at "O".

At O, introduce two equal and opposite forces of magnitude 2P along EF, parallel to the forces P at A and B. By this, the effect of the given couple is not altered.

Now the unlike parallel forces P along AL and 2P along OE can be compounded into a single force P acting at D,

Since $\frac{AD}{OD} = \frac{2}{1} = \frac{2P}{P}$

This resultant force P acts along DN in the second plane similarly the unlike parallel forces P along BM and 2P along OF can be compounded into a single force P acting at C along CK, We are therefore left with a couple of forces P at the ends of the arm CD in a plane parallel to that of the original couple.

Thus, the given couple with the arm AB is equivalent to another couple of the same moment in a parallel plane, having its arm CD equal and parallel to AB.

Now this couple with arm CD can be replace in its own plane by another couple, provided the moment is unchanged in magnitude and direction.

Hence we conclude that a couple in any plane can be replaced by another couple acting in a parallel plane, Provided that the moment of the two couples are the same in magnitude and direction.

Representation of a couple by a vector

A couple is completely specified if we know i) The direction of the set of Parallel sence in which it acts. These three aspects of a couple can be conveniently represented by a straight line drawn.

i) Perpendicular to the set of parallel Planes to indecate the direction, ii). Of a measured length, to indicate the moment of the couple and (iii) In a definite direction, to indecate the sence of the moment.

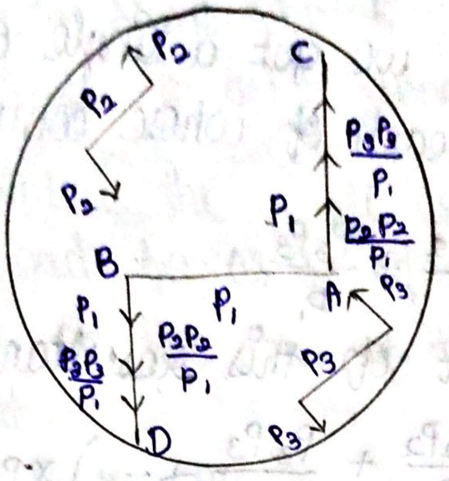
Such a vector which is drawn to represent a couple is called the axis of the couple.

Resultant of plane couples

Theorem:-

The resultant of any number of couples in the same plane on a riged body is a single couple whose moment is equal to the algebraic sum of the moments of several couples.

Proof :-



Let (P_1,P_1), (P_2,P_2), (P_3,P_3)....etc.... be a number of couples acting in the same plane upon a body. Let AB represent the arm p_1 of the first couple (P_1,P_1) whose moment forces P_1 act along AC and BD.

The moment of the second couple $(P_2,P_2) = P_2 p_2$

This couple canbe replaced by an equivalent couple, having its arm along AB and having its forces along AC and BD.

If F is the forces of such a replacing couple we have

$$F \cdot p_1 = P_2 p_2$$

$$F = \frac{P_2 p_2}{p_1}$$

Thus the couple (P_2,P_2) is replaced by another couple whose arm coincides with AB and whose component forces along AC and BD are of magnitude $\frac{P_2 p_2}{p_1}$

Similarly the couple (P_3,P_3) is replaced by a couple $\left(\frac{P_3 p_3}{p_1}, p_1\right)$ with the forces $\frac{P_3 p_3}{p_1}$ along AC and BD. This process is replaced for other couples.

Finally we get a single couple with the arm AB, each of whose component forces

$$= P_1 + \frac{p_2 P_2}{p_1} + \frac{p_3 P_3}{p_1} + \dots$$

The moment of this resultant couple

$$= \left(P_1 + \frac{p_2 P_2}{p_1} + \frac{p_3 P_3}{p_1} + \dots\right) \times p_1$$

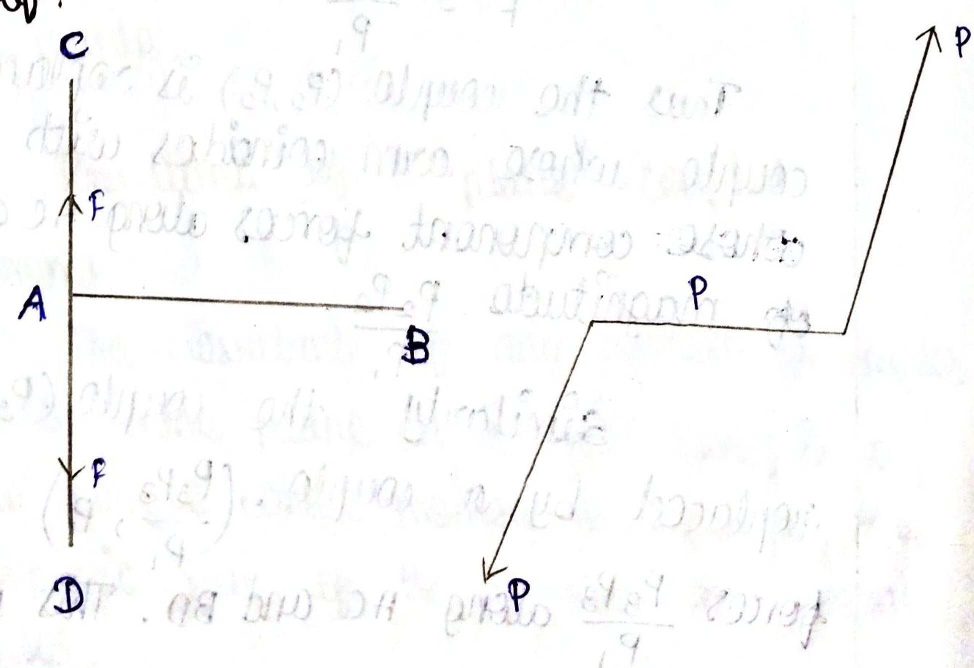
$$= P_1 p_1 + P_2 p_2 + P_3 p_3 + \dots$$

= The algebraic sum of the moments of the several couples.

Resultant of a couple and a force.

Theorem:- A couple and a single force acting on a body cannot be in equilibrium but the are equivalent to the single force acting at same other point parallel to its original direction.

Proof:-



Let the given couple (P,P) by another and the given force be F lying in the same plane. Let F act along AC.

Replace the couple (P,P) by another couple whose each forces is equal to F. If x be the length of the arm of this new couple, its moment

$$= F.x = Pp$$

$$\Rightarrow x = \frac{Pp}{F}$$

Place this couple such that one of its component forces F acts at A along the line of acting action of the given force F but it the opposite direction. i.e., it acts along AD

The original force F along AC and the force F along AD balance.

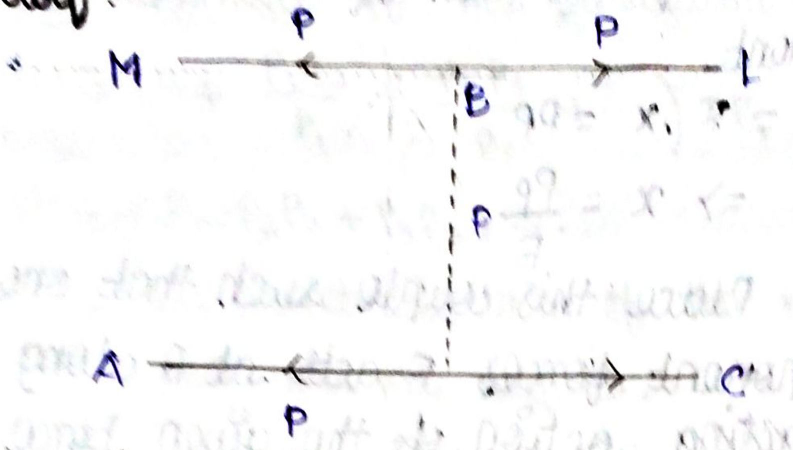
We are left a force F acting at B Parallel to AC, as the statical equivalent of the system.

$$\text{Also } AB = x = \frac{Pp}{F}$$

Hence the couple (P,P) and the forces F are equivalent to an equal force F. Parallel to its original direction at a distance $\frac{Pp}{F}$ from its original line of action.

Theorem: A force acting at any point A of a body is equivalent to an equal and parallel force acting at any other arbitrary point B of the body, together with a couple.

Proof:-



Let P be a force acting at A along AC and B any arbitrary point. Let p be the distance of B from AC.

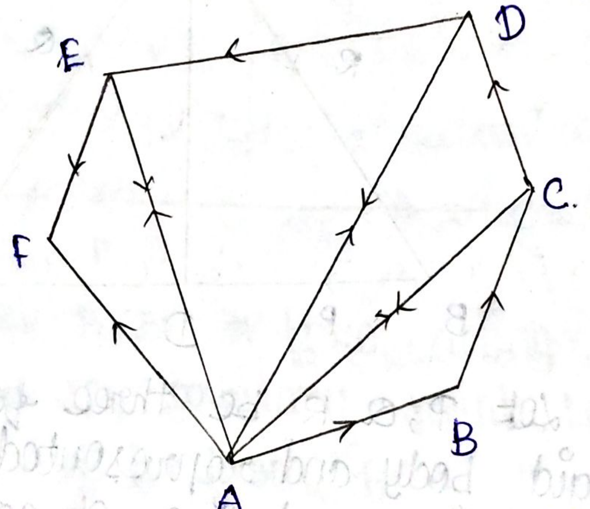
At B, apply two equal and opposite forces each equal and parallel to P along BL and BM. These two new forces being equal and opposite, will have no effect on the body. Of the three forces now acting on the body, the forces P along BM and P along AC, form a couple and the remaining is the force P acting at B, parallel to the original force.

Thus the statical equivalent of the original force P at A is an equal and parallel force P at B, together with a couple whose moment is Pp, where p is the perpendicular distance of B from AC.

being two equal and opposite forces from a couple whose moment $= P \cdot AD = BC \cdot AD$
$= 2\,\Delta ABC$.

Theorem:- If any number of forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a polygon taken in order, they are equivalent to a couple whose moment is twice the area of the polygon.

Proof:-



Let the forces be represented completely by the sides AB, BC, CD, DE, EF and FA of the closed Polygon ABCDEF. Join AC, AD and AE.

Introduce along AC, AD and AE, pairs of equal and opposite forces represented completely by those lines. These new forces do not affect the resultant of the system.

$\therefore$ By Theorem.

$\overline{AB} + \overline{BC} + \overline{CA}$ = a couple whose moment is equal to $2\,\Delta ABC$.

$\overline{AC} + \overline{CD} + \overline{DA}$ = a couple whose moment is equal to $2\,\Delta ACD$.

$\overline{AD} + \overline{DE} + \overline{EA}$ = a couple whose moment is equal to $2\Delta ADE$

$\overline{AE} + \overline{EF} + \overline{FA}$ = a couple whose moment is equal to $2\Delta AEF$

Adding vectionally,

$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF} + \overline{FA}$ = resultant of four couples.

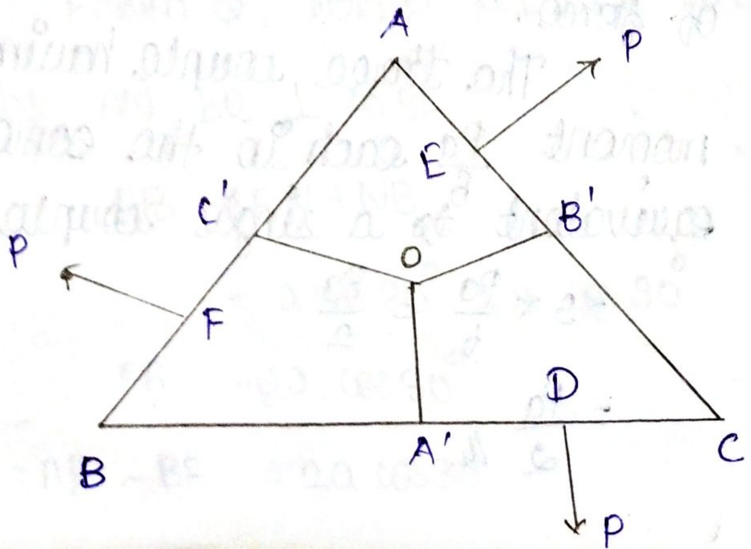
= a single couple whose moment is equal to $2(\Delta ABC + \Delta ACD + \Delta ADE + \Delta AEF)$.

ie, The resultant is a couple whose moment is equal to twice the area of the Polygon ABCDEF.

Problems:-

12. ABC is an equilatual, triangle of side 'a'. D,E,F divide the sides BC, CA, AB respectively in the ratio 2:1. Three forces each equal to P act at D,E,F Perpendicularly to the sides and out coard from the triangle prove that they are equivalent to a couple of moment $\frac{1}{2}$ Pa.

Solns:-



Let O be the circumcentre of the equilateral triangle ABC and A', B', C' be the middle points of the sides.

OA' is $\perp_r$ to BC.

$\therefore$ By theorem, the force P acting at D, $\perp_r$ to BC is equivalent to a parallel force P acting at O along OA' together with a couple whose moment.

$$= P \cdot A'D = P(A'C - DC)$$

$$= P \cdot \left(\frac{a}{2} - \frac{a}{3} \right) = \frac{Pa}{6}$$

Similarly the force P acting at E $\perp r$ to CA is replaced by a parallel force P acting at O along OB' together with a couple whose moment $= \frac{Pa}{6}$

The force P acting at F $\perp r$ to AB is replaced by a parallel force P acting at O along OC' together with a couple whose moment $= \frac{Pa}{6}$

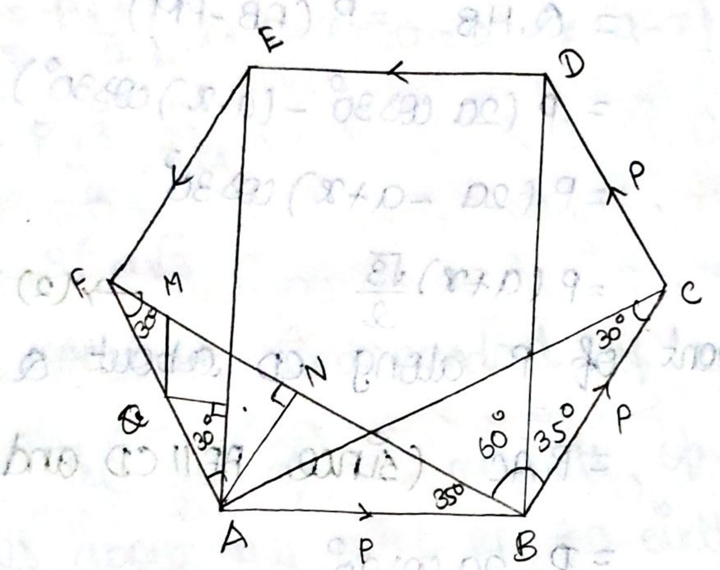
The three equal force P acting at O $\perp r$ to the sides of the triangle are in equilibrium by the perpendicular triangle of force.

The three couple having the same moment $\frac{Pa}{6}$ each in the same direction are equivalent to a single couple whose moment

$$= 3 \times \frac{Pa}{6} = \frac{Pa}{2}$$

$$= \frac{Pa}{2}$$ H.

2. Five equal forces act along the sides AB, BC, CD, DE, EF of a regular hexagon. Find the sum of the moments of these forces about the point Q on AF at a distance x from A. Interpret the result and explain why it is so.



Let "a" be the length of each side of the regular hexagon. Each interior angle of the regular hexagon $= 120^\circ$

We know that $AB \parallel DE$, $BC \parallel EF$ and $DC \parallel AF$; $FB \perp BC$, $AF \perp AB$, $DB \perp AB$

Let equal force P act along the sides AB, BC, CD, DE, and EF. Q is a point on AF such that $AQ = x$.

From Q, draw $QL \perp EA$ and $QH \perp BF$

Let AN be $\perp$ to BF

$FB = FN + NB$

$= a\cos 30^\circ + a\cos 30^\circ$

$FB = 2a\cos 30^\circ$

$AC = AE - BF = 2a\cos 30^\circ$

Moment of P along AB about Q

$= P.AL = P x \cos 30^\circ$ (From right angle ΔAQL).

$= P x \frac{\sqrt{3}}{2}$ $\rightarrow (1)$

Moment of P along BC about Q

$= Q.HB = P(FB - FM)$

$= P(2a \cos 30^\circ - (a-x)\cos 30^\circ)$

$= P(2a - a + x)\cos 30^\circ$

$= P(a+x)\frac{\sqrt{3}}{2}$ $\rightarrow (2)$

Moment of P along CD about Q.

$= P.AC$ (since $AF \parallel CD$ and $AC \perp CD$).

$= P.\ 2a \cos 30^\circ$

$= P.\ 2a \frac{\sqrt{3}}{2}$

$= Pa\sqrt{3}$ $\rightarrow (3)$.

Moment of P along DE about Q

$= P.EL$

$= P(AE - AL)$

$= P(2a \cos 30^\circ - x \cos 30^\circ)$

$= P(2a - x)\cos 30^\circ$

$= P(2a - x)\frac{\sqrt{3}}{2}$ $\rightarrow (4)$

Moment of P along EF about Q

$= P.MF$

$= P(a-x)\cos 30^\circ$

$= P(a-x)\frac{\sqrt{3}}{2}$. $\rightarrow (5)$

Adding up, the sum of the moments of the five forces about Q

$$= P x\frac{\sqrt{3}}{2} + P(a+x)\frac{\sqrt{3}}{2} + Pa\sqrt{3} + P(2a-x)\frac{\sqrt{3}}{2} + P(a-x)\frac{\sqrt{3}}{2}$$

$$= P\frac{\sqrt{3}}{2}[x+a+x+2a+2a-x+a-x]$$

$$= P\frac{\sqrt{3}}{2}.6a$$

$$= 3Pa\sqrt{3}$$

= constant independent of x

The sum of the moments of the five forces about any point on the sixth side AF is a constant.

Introduce two equal and opposite forces, each to P along the sixth side. These new forces donot affect the resultant of the system. We have now seven forces; The moment of the new force P introduced along AF about Q is $= 0$

The other six forces act along the sides of the hexagon and are represented in magnitude, direction and line of action by the sides of the hexagon.

Hence by theorem, they are equivalent to a couple whose moment is

= 2 x Area of the hexagon.

$$= 2 \times 6a^2 \frac{\sqrt{3}}{4}$$

$$= 3a^2\sqrt{3}$$

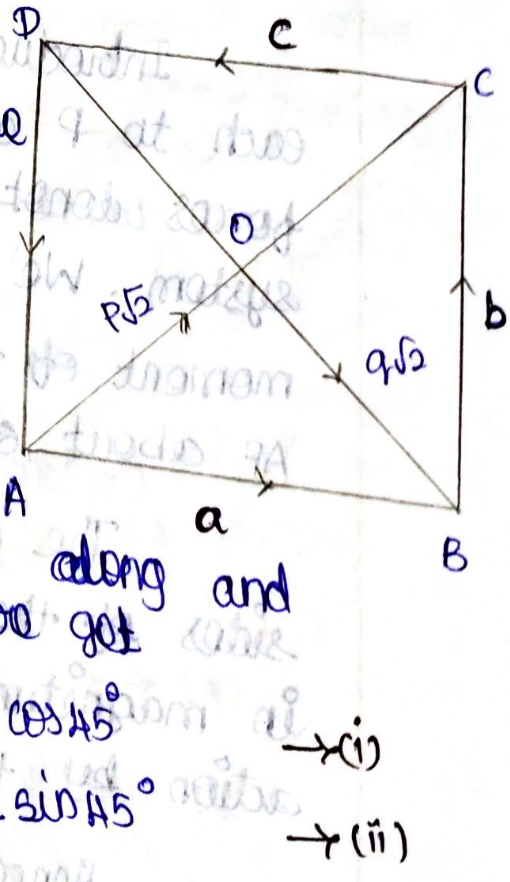
$$= 3a\sqrt{3} \cdot a$$

$= 3a\sqrt{3} \cdot P$ [an P is represented in magnitude by a]

3. ABCD is a square whose side is 2 units in length. Forces a, b, c, d, act along the sides AB, BC, CD and DA taken in order and $P\sqrt{2}$, $q\sqrt{2}$ act along AC and DB respectively. Show that if $P+q = c-a$, $P-q = d-b$, the forces are equivalent to a couple of moment $a+b+c+d$.

Soln:-

If the given forces are equivalent to a couple the sum of the resolved parts of all forces in any direction must be zero.



Resolving the forces along and perpendicular to AB, we get

$$a - c + P\sqrt{2} \cdot \cos 45^\circ + q\sqrt{2} \cos 45^\circ \rightarrow (i)$$

and

$$b - d + P\sqrt{2} \cdot \sin 45^\circ - q\sqrt{2} \sin 45^\circ \rightarrow (ii)$$

If $c - a = P+q$ and $d-b = P-q$

Then from (i) and (ii) we get that the sum of the resolved parts of the two forces in two directions are zero. Hence the forces are equivalent to a couple.

The moment of the couple about O = algebraic sum of the moments of forces constituting the couple about O

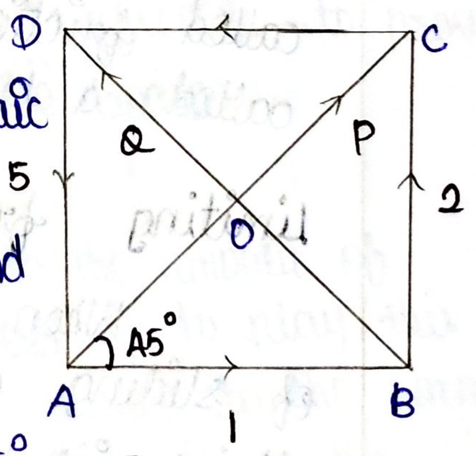
$= a \cdot 1 + b \cdot 1 + c \cdot 1 + d \cdot 1$

$= a + b + c + d$

A. Forces 1, 2, 3, 5, P, Q act along AB, BC, CD, DA, AC, BD respectively and ABCD is a square. Find the values of P and Q for the system to reduce to a couple. Find also the moment of the couple.

Soln:

The system reduces to a couple only if the resultant forces is zero. Hence the algebraic sum of the resolved parts of the forces along AB and AD must vanish.



$1 - 3 + P\cos 45^\circ - Q\cos 45^\circ = 0$

and $\therefore P - Q = 2\sqrt{2}$

$2 - 5 + P\sin 45^\circ + Q\sin 45^\circ = 0$

(or) $\therefore P + Q = 3\sqrt{2}$

Solving $P = \frac{5\sqrt{2}}{2}$, $Q = \frac{\sqrt{2}}{2}$

Moment of the couple

= algebraic sum of the moments of the forces about O

$= 1 \cdot a/2 + 2 \cdot a/2 + 3 \cdot a/2 + 5 \cdot a/2$

$= 11 \cdot a/2$, a being each side.