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COMPLEX ANALYSIS

B.Sc. MATHEMATICS

V - SEMESTER

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UNIT - II

INTEGRALS

Derivative of function $w(t)$:-

Consider the derivatives of complex valued function w of a real variable.

ie., $w(t) = u(t) + iv(t) \rightarrow (1)$

where the functions u and v are real values functions of t.

The derivative $w'(t)$ (or) $\frac{d}{dt} w(t)$ of the function eqn (1) at a point t is defined

$$w'(t) = u'(t) + i v'(t) \rightarrow (2)$$

each of the derivatives u' and v' at t

Now, we know that

$z = x + iy$ and every complex constant,

$$z_0 = x_0 + iy_0$$

so, (2) $\Rightarrow w'(t) = u'(t) + iv'(t)$

$$\frac{d}{dt}[z_0 w(t)] = [(x_0 + iy_0)(u + iv)]'$$

$$= [x_0 u + ivx_0 + iy_0 u - vy_0]'$$

$$= [(x_0 u - y_0 v)' + i(y_0 u + x_0 v)]$$

$= [(x_0u' - y_0v') + i(y_0u' + x_0v')]$

$= (x_0 + iy_0)u' + (x_0 + iy_0)v'$

$= (x_0 + iy_0)(u' + iv')$

$= z_0 w'(t)$

$\frac{d}{dt}[z_0 w(t)] = z_0 w'(t) \rightarrow (3)$

Another Rule :-

$\frac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t} \rightarrow (4)$

where, $z_0 = (x_0, y_0)$

ie, $z_0 = x_0 + iy_0$

$e^{z_0 t} = e^{x_0 t} e^{iy_0 t}$, $e^{(x_0+iy_0)}$

$= e^{x_0 t}(\cos y_0 t + i \sin y_0 t)$

$= e^{x_0 t}\cos y_0 t + ie^{x_0 t}\sin y_0 t$

From (2),

$w'(t) = u'(t) + iv'(t)$

$\frac{d}{dt} e^{z_0 t} = (\underset{u}{e^{x_0 t}}\cos y_0 t)' + i(e^{x_0 t}\sin y_0 t)'$

$v: uv' + vu'$

$\frac{d}{dt} e^{z_0 t} = (\underset{u}{x_0 e^{x_0 t}}\cos y_0 t + \underset{v}{e^{x_0}} - \sin y_0 t \cdot y_0) + i(x_0 e^{x_0 t}\sin y_0 t + e^{x_0 t}\cos y_0 t \cdot y_0)$

$d(uv) = uv' + vu'$

$= x_0 e^{x_0 t}\cos y_0 t - e^{x_0 t} y_0 \sin y_0 t + ix_0 e^{x_0 t}\sin y_0 t + ie^{x_0 t}\cos y_0 t\, y_0$

$= e^{x_0 t}\cos y_0 t\,[x_0 + iy_0] + e^{x_0 t}\sin y_0 t\,[ix_0 - y_0]$ $i(x_0 + y_0)$ $x_0 + iy_0$

$= e^{x_0 t}\cos y_0 t\,[x_0 + iy_0] + e^{x_0 t}\sin y_0 t\,[x_0 + iy_0]$

$= [x_0 + iy_0][e^{x_0 t}\cos y_0 t + ie^{x_0 t}\sin y_0 t]$

$\frac{d}{dt} e^{z_0 t} = [x_0 + iy_0]\, e^{x_0 t}\,[\cos y_0 t + i\sin y_0 t]$

$= [x_0 + iy_0]\, e^{x_0 t} \cdot e^{iy_0 t} \rightarrow (5)$

Hence eqn (4) & (5) are same equation.

Ex:-

Suppose that $w(t)$ is continuous on an interval $a \le t \le b$.

ie, the component function $u(t)$ and $v(t)$ are continuous there. w', w'', w'''

Then $w'(t)$ exist when $a < t < b$. The the mean value theorem for derivatives no longer applies,

Now, introduce a number c in the interval $a < t < b$ such that,

$$w'(c) = \frac{w(b) - w(a)}{b-a}$$

consider the function, $w(t) = e^{it}$ on the interval $0 \le t \le 2\pi$.

$$w(t) = e^{it}$$

$$w'(t) = i e^{it}$$

$\Rightarrow |w'(t)| = |ie^{it}| = 1$, and this mean the derivatives $w'(t)$ is never zero.

while $w(2\pi) - w(0) = 0$

Definite Integrals of function $w(t)$:

When $w(t)$ is a complex valued function of a real variable t and is written,

$$w(t) = u(t) + iv(t) \rightarrow ①$$

where u, v are real valued, The definite integrals $w(t)$ over an interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \rightarrow ②$$

provided the individual integrals on the right exists. Thus,

$$\text{Re} \int_a^b w(t)\,dt = \int_a^b \text{Re}\,[w(t)]\,dt$$

$$\text{Im} \int_a^b w(t)\,dt = \int_a^b \text{Im}\,[w(t)]\,dt$$

For example of definition ②, $\int_0^1 (1+it)^2\,dt$

Sol:- $\int_0^1 (1+it)^2 dt = \int_0^1 (1-t^2+2it)\,dt$

we know that,

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_0^1 v(t)\,dt$$

$$\Rightarrow \int_0^1 (1+it)^2\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt$$

$$= \left[t - \frac{t^3}{3}\right]_0^1 + i\left[\frac{2t^2}{2}\right]_0^1$$

$$= (1-1/3) + i(1)$$

$$= \frac{3-1}{3} + i$$

$$\therefore \int_0^1 (1+it)^2\,dt = 2/3 + i$$

Note:

The existence of the integrals of u and v in eqn no(2) is ensured if those function are piecewise continuous on the interval $a \le t \le b$ such a function is continuous every where in the state interval expect possible for a finite no. of points where another discontinuous. it has one side limits.

The right hand limit of required at and only the left hand limit is required at b when both a and v are piecewise continuous the function w said to have that property.

Interchanging the limit :-

Integrating a complex constant times a function w(t), the integrating some of such functions and for interchanging limits of integration are all valued.

ie, $\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$

Remark :

The fundamental theorem of columns involving antiderivatives be extented so to apply two integrals of

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Now,

Suppose that the function $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ are continuous on the interval $a \le t \le b$. If $W'(t) = w(t)$ when $a \le t \le b$ then,

$$U'(t) = u(t) \text{ and } V'(t) = v(t)$$

$$\therefore \int_a^b w(t)\,dt = \int_a^b u(t)\,dt + iv(t)\,dt$$

$$= \int_a^b U(t) + i\int_a^b V(t)$$

$$= [U(b) - U(a)] + i[V(b) - V(a)]$$

$$= [U(b) - U(a) + iV(b) - iV(a)]$$

$$= [U(b) + iV(b)] - [U(a) + iV(a)]$$

$$\therefore \int_a^b w(t)\,dt = W(b) - W(a)$$

ie $\int_a^b w(t)\,dt = \int_a^b w(t)\,dt.$

contours :-

A set of points $z=(x,y)$ in the complex plane is said to be an arc. If $x=x(t), y=y(t)$ on $(a \leq t \leq b)$. where $x(t)$ and $y(t)$ are continuous function of the real parameter t.

The arc c is a simple arc (or) a Jordan arc. if it does not cross itself.

ie., c is a simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$

when the arc c is simple except for the fact $z(b)=z(a)$ then we say the c is a simple closed curve (OR) a Jordan curve. Such a curve is positively oriented when it is in the countor clase clock wise direction.

Note:

To describe the point of c by means of the equation $z=z(t)$ on $a \leq t \leq b$. where $z(t)=x(t)+iy(t)$.

Examples :-

1) The polygonal line defined by means of the equation

$$z = \begin{cases} x+ix & \text{when } 0 \leq x \leq 1 \\ x+i & \text{when } 1 \leq x \leq 2 \end{cases}$$

whether it is simple arc (or) simple closed curve.

Sol :-

1) Given, $z = \begin{cases} x+ix & \text{when } 0 \leq x \leq 1 \\ x+i & \text{when } 1 \leq x \leq 2 \end{cases}$

$z = x+ix$ when $0 \leq x \leq 1$

$x=0 \Rightarrow z = 0+i(0) = 0$

$x=1 \Rightarrow z = 1+i(1) = 1+i$

$z(0) \neq z(1)$

$\therefore$ It is a simple arc.

ii) $z = x+i$ when $1 \leq x \leq 2$

$x=1 \Rightarrow z = 1+i$

$x=2 \Rightarrow z = 2+i$

$z(1) \neq z(2)$

$\therefore$ It is a simple arc.

2. The unit circle $z = e^{i\theta}$ on the intervals $0 \leq \theta \leq 2\pi$.

Sol:-

Given, $z = e^{i\theta}$ on $0 \leq \theta \leq 2\pi$

i) $\theta = 0 \Rightarrow z^{i(0)} = 1$

ii) $\theta = 2\pi \Rightarrow z = e^{i(2\pi)}$

$= \cos 2\pi + i \sin 2\pi$

$z = 1+0$

$z = 1$

$\therefore z(0) = z(2\pi)$

$\therefore$ It is simple closed curve and it is oriented in the countor clockwise direction.

3) $z = e^{i2\theta}$ on the interval $0 \leq \theta \leq 2\pi$

Sol!-

Given $z = e^{i2\theta}$ on $0 \leq \theta \leq 2\pi$

i) $\theta = 0$

$z = e^{i2(0)} = e^0 = 1$

$z = 1$

ii) $\theta = 2\pi$

$z = e^{i2(2\pi)} = e^{i4\pi}$

$= \cos 4\pi + i \sin 4\pi$

$= 1+0$

$= 1$

$z(0) = z(2\pi)$

$\therefore$ It is simple closed curve and it is oriented in the twice of countor clockwise.

<u>Differentiable arc</u> :

Suppose that the components of $x'(t)$ and $y'(t)$ of the derivatives $z'(t)$

i.e., $z'(t) = x'(t) + iy'(t)$

It is used to represent C are continuous on the entire interval $a \leq t \leq b$. The arc is then called a differentiable arc and the real valued function

$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ is integrable over the interval $a \leq t \leq b$.

<u>Smooth curve</u> :-

A curve or an arc is said to be a smooth then the following conditions are satisfied.

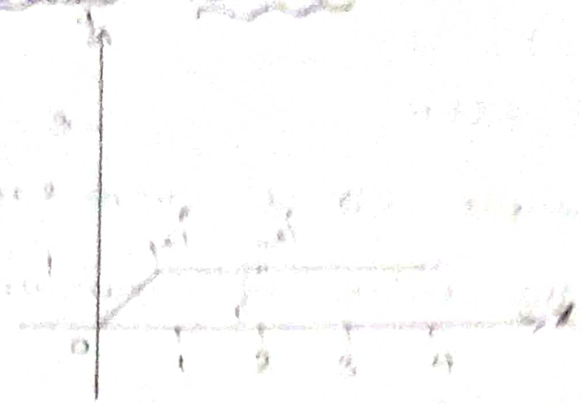
1) $z(t)$ has a continuous derivative on the interval $[a,b]$.

ii) $z'(t)$ is never zero on open interval (a,b).

iii) $z(b)$ is a one to one function on $[a,b]$.

Hence, the first two conditions are not but $z(a) = z(b)$ is a smooth closed curve.

In geometric nature :-



Jordan curve theorem (statement) :-

The points on any simple closed curve or simple closed contour c are boundary points of two distinct domains one of which is the interior of c and is bounded. The other which is exterior of c is unbounded.

Contour Integrals :-

Integration of a complex function of a complex variable is performed on a set of connected point from say z_1 to z_2 it is a contour integral

Given a contour c defined as $z(t)$ for $a \le t \le b$, where $z_1 = z(a)$ and $z_2 = z(b)$ an integral of a complex function of a complex variable $f(z)$ is written as.

$$\int_c f(z)\, dz \text{ (or) } \int_{z_1}^{z_2} f(z)\, dz$$

let $f(z)$ be piecewise continuous on $z(t)$ if c is contour then $z'(t)$ is piecewise continuous on $a \le t \le b$ and we can defined the integral of $f(z)$ along c as $\int_C f(z) dz = \int_a^b f[z(t)] z'(t)\, dt$.

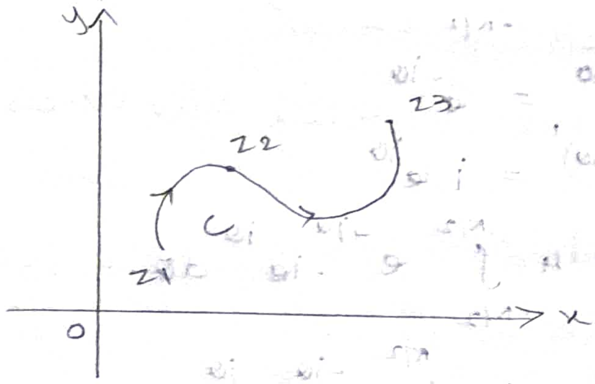
Properties of contour integral :-

Suppose the function f and g are continuous complex function of a complex variable in a domain D and c is a piece wise smooth curve lying anti derivative in D then,

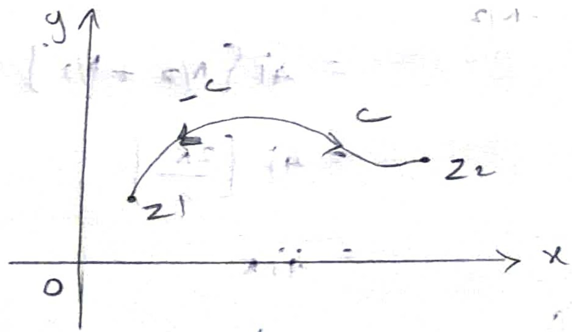
i) $\int_c z_0 f(z)\, dz = z_0 \int_c f(z)\, dz$ where z_0

ii) $\int_c [f(z)+g(z)]\, dz = \int_c f(z)\, dz + \int_c g(z)\, dz$

iii) $\int_c f(z)\, dz = \int_{c_1} f(z)\, dz + \int_{c_2} f(z)\, dz$ where c would be formed from joining c_1 and c_2 and to end.



iv) $\int_{-c} f(z)\, dz = -\int_c f(z)\, dz$ where $-c$ has the opposite orientation of c.



(Some points but order reversed)

$z_2 \rightarrow z_1 \Rightarrow -c$

Some examples of properties :-

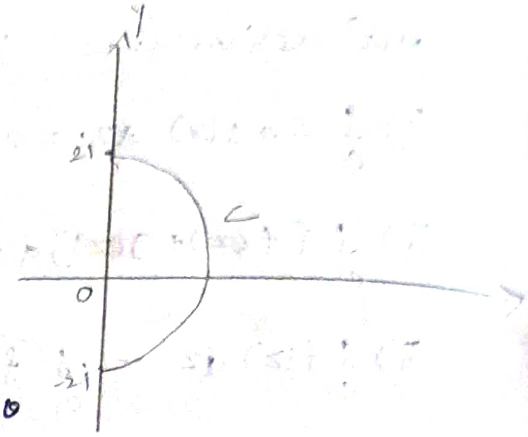
1) Find the value of the integral $I = \int_c \bar{z}\, dz$ when c is the right hand half $z = 2e^{i\theta}$ $\left(-\frac{\pi}{2} \le \theta \le \pi/2\right)$ of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$

Sol :-

Given $z = 2e^{i\theta}$ $\left(-\frac{\pi}{2} \le \theta \le \pi/2\right)$ of the circle.

To find the line integral $I = \int_C \bar{z}\, dz$ by using the properties of contour integration.

Now, we know that



$$\int_C f(z)\, dz = \int_a^b f[z(t)]\, z'(t)\, dt$$

So given

$$z = 2e^{i\theta} \left[-\frac{\pi}{2} \le \theta \le \pi/2\right]$$

$$I = \int_C \bar{z}\, dz = \int_{-\pi/2}^{\pi/2} 2\overline{e^{i\theta}} \cdot (2e^{i\theta})'\, d\theta$$

$$= 4 \int_{-\pi/2}^{\pi/2} \overline{e^{i\theta}}\, (e^{i\theta})'\, d\theta$$

i.e., $\overline{e^{i\theta}} = e^{-i\theta}$

$(e^{i\theta})' = i e^{i\theta}$

$$\therefore I = 4 \int_{-\pi/2}^{\pi/2} e^{-i\theta} \cdot i e^{i\theta}\, d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} e^{-i\theta} \cdot e^{i\theta} \cdot d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} d\theta = 4i\, [\theta]_{-\pi/2}^{\pi/2}$$

$$= 4i\, [\pi/2 + \pi/2]$$

$$= 4i \left[\frac{2\pi}{2}\right]$$

$$= 4i\pi$$

Hence, $\int_C \bar{z}\, dz = 4\pi i$

2. Evaluate the integral $I_1 = \int_{C_1} f(z) dz = \int_{OA} f(z) dz + \int_{AB} f(z) dz$ where $f(z) = y - x - i3x^2$ then C_1 denote the polygonal the OABO

(or)

Evaluate $\int_C f(z) dz$ where C is a simple closed contour OABO, $f(z) = y - x - i3x^2$ of OABD $= C_1$

(or)

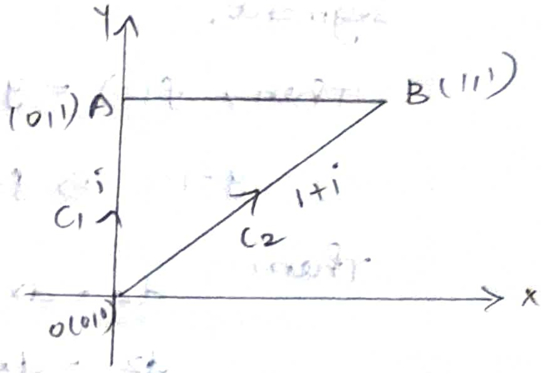
P.T $\int_C f(z)\,dz = -\frac{1+i}{2}$ where C is a simple closed contour OABD with $f(z) = y - x - i3x^2$ $(z = x+iy)$

simple closed contours [SCC]:-

A contour is a piecewise smooth curve.

ie) $z(t)$ is continuous but $z'(t)$ is only piecewise continuous.

If the initial and final values of $z(t)$ are the same $z(a) = z(b)$ a contour is called simple closed contours.



Sol:-

Let C_1 denote the polygonal line OAB shown in figure.

$$I_1 = \int_{C_1} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz \rightarrow (A)$$

Given that,

$f(z) = y - x - i3x^2$, where $z = x+iy$.

i) Now,

the leg OA may be represented parametrically

$z = x+iy$

ie, $z = 0+iy$ $(0 \le y \le 1)$

Since, $x=0$ at the points on that line segment, the values of f there vary with the parameter y. According to the equation,

$$f(z) = y - x - 13x^2$$

$$x = 0 \Rightarrow f(z) = y \quad (0 \le y \le 1)$$

Then $dz = dx + i\,dy$

$dx = i\,dy$

$$\therefore \int_{OA} f(z)\,dz = \int_0^1 iy\,dy$$

$$= i \int_0^1 y\,dy$$

$$= i\left[\frac{y^2}{2}\right]_0^1 = i\left[\frac{1}{2}\right]$$

$$\int_{OA} f(z)\,dz = \frac{i}{2} \rightarrow (1)$$

ii) Next on the log AB, the points are (1,1)

$z = x+iy$

$z = x+i \quad (0 \leq x \leq 1)$ since $y=1$ on this segment.

Then, $f(z) = y - x - i3x^2$

$y=1 \Rightarrow f(z) = 1 - x - i3x^2 \quad (0 \leq x \leq 1)$

Then, $dz = dx + i\,dy$

$dz = dx + 0$

$dz = dx \qquad y=1$

$dy = 0$

$$\therefore \int_{AB} f(z)\,dz = \int_0^1 (1 - x - i3x^2)\,dx \quad \| y=1$$

$$= \left[x - \frac{x^2}{2} - i\frac{3x^3}{3}\right]_0^1$$

$$= \left[x - \frac{x^2}{2} - ix^3\right]_0^1$$

$$= \left[1 - \tfrac{1}{2} - i\right] - 0$$

$$\int_{AB} f(z)\,dz = \tfrac{1}{2} - i \rightarrow (2)$$

(A) $\Rightarrow I_1 = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz$

$I_1 = \frac{i}{2} + \frac{1}{2} - i = \frac{i+1-2i}{2}$

$I_1 = \frac{1-i}{2} \rightarrow (3)$

ii) next, C_2 denotes the segment OB of the line $y = x$ with parametrix $z = x + ix$ $(0 \le x \le 1)$

ie, $z = x + ix$

$= x(1+i)$

w.r.t,

$dz = dx + i\,dx$

$dz = (1+i)\,dx$

Then,

$f(z) = y - x - i3x^2$

$f(z) = x - x - i3x^2$

$f(z) = -i3x^2$

$$I_2 = \int_{C_2} f(z)\,dz$$

$$= \int_0^1 -i3x^2 (1+i)\,dx$$

$$= \int_0^1 (-i3x^2 + 3x^2)\,dx$$

$$= \int_0^1 3x^2 (1-i)\,dx$$

$$= 3(1-i) \int_0^1 x^2\,dx$$

$$= 3(1-i) \left[\frac{x^3}{3}\right]_0^1$$

$$= 3(1-i)(1/3)$$

$$I_2 = \frac{(1-i)3}{3}$$

$$I_2 = (1-i)$$

$$\therefore I_2 = \int_{OB} f(z)\,dz = 1 - i \rightarrow (4)$$

From this, the integrals of $f(z)$ along to two paths C_1 and C_2 have different values even though path have the same initial and final points.

From the follows that the integrals of $f(z)$ is simple closed contour OABO (or) $C_1 - C_2$ has the non zero value.

ie, $I_1 - I_2 = \frac{1-i}{2} - (1-i)$

$$= \frac{1-i-2(1-i)}{2}$$

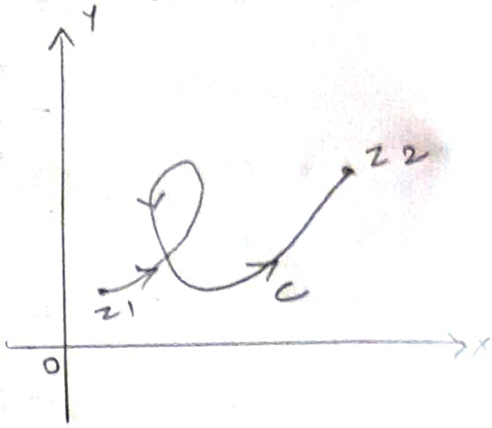
$$= \frac{1-i-2+2i}{2}$$

$$I_1 - I_2 = \frac{-1+i}{2}$$

3) Evaluate (or) prove that $\int_C f(z)dz$ where C is an arbitrary smooth arc at $z = z(t)$ on interval $a \le t \le b$ from a point z_1 to a fixed point z_2.

Sol:-

Given that $z = z(t)$ on $a \le t \le b$ from a fixed point z_1 to a fixed point.



To prove that, C is an arbitrary smooth arc.

we know that,

The property of contour Integrals

$$\int_C f(z)dz = \int_a^b f(z)\, f'(z)\, dz$$

ie, $$\int_C z\, dz = \int_a^b z(t)\, z'(t)\, dt \rightarrow (1)$$

Already we know that,

$$\frac{d}{dt}[z(t)]^2 = 2\, z(t) \cdot z'(t)$$

$$\Rightarrow \frac{d}{dt} \frac{[z(t)]^2}{2} = z(t) \cdot z'(t) \rightarrow (2)$$

They, $z(a) = z_1$ and $z(b) = z_2$ we have

$$\int_C z\, dz = \int_a^b \frac{[z(t)]^2}{2}$$

$$= \frac{[z(b)]^2 - [z(a)]^2}{2}$$

$$= \frac{z_2^2 - z_1^2}{2}$$

So, $$\int_{z_1}^{z_2} z\, dz = \frac{z_2^2 - z_1^2}{2} \rightarrow (3)$$

From (3) $\Rightarrow$ when c is a contour that is not necessarily smooth, since a contour consists of a finite number, of smooth arcs

$C_k : (k = 1, 2, \ldots n)$

joined and to end. So, we C_k extends from z_k to z_{k+1}.

$$\text{Then, } \int_c z\, dz = \sum_{k=1}^{n} \int_{c_k} z\, dz$$

$$= \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} z\, dz$$

$$= \sum_{k=1}^{n} \frac{z^2_{k+1} - z_k^2}{2}$$

$$= \frac{z_{k+1}^2 - z_k^2}{2}$$

$$\therefore \int_c z\, dz = \frac{z^2_{n+1} - z_n^2}{2}$$

i.e., $z_1 \rightarrow$ Initial point

$z_{n+1} \rightarrow$ Final point

Contour:

A contour or piecewise smooth arc, is arc consisting of a finite number of smooth arc joined end to end.

Branch cut:

Branch cut is a curve with ends possible open, closed or half open in the complex plane acorss with an analytic multi value function is discontinuous.

The path in a contour integral can contain a point on a branch cut.

Examples with branch cut :-

1) Evaluate $\int_C f(z)\,dz$ is a branch cut from $z = 3e^{i\theta}$ on interval $0 \le \theta \le \pi$ from the point $z = 3$ to $z = -3$ Then c is denote the semicircular path.

(or)

Find the contour integral from the branch cut of $\int_C f(z)\,dz$ in $z = 3e^{i\theta}$ on $0 \le \theta \le \pi$ from the point $z = 3$ to $z = -3$ 'c' is denote the semi circular path.

Repeated

Sol:-

Given $\int_C f(z)\,dz$ is a branch cut.

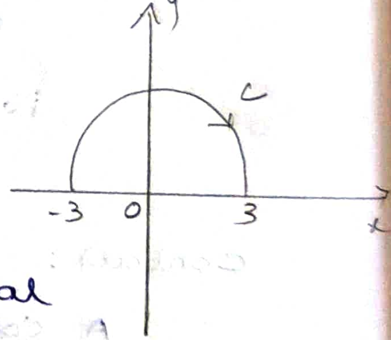
Then,

$$z = 3e^{i\theta} \text{ on the interval } 0 \le \theta \le \pi$$

$$\therefore f(z) = z^{1/2} \qquad \because m^n = n\log m$$

$$= \exp\left(\tfrac{1}{2}\log z\right)$$

$$(|z| > 0,\ 0 < \arg z < 2\pi)$$



Of multiple - valued function $z^{1/2}$ is not defined at the initial point

i) $z = 3$ of the contour c

Then, the integral becomes,

$$I = \int_C f(z)\,dz$$

$$= \int_C z^{1/2}\,dz$$

ie, Integral is piecewise continuous on c.

$z(\theta) = 3e^{i\theta}$

$z(\theta)^{1/2} = (3e^{i\theta})^{1/2}$

$= [1 - [3e^{i\theta}]^{1/2}]$

$= \exp \frac{1}{2} \{ \log (3e^{i\theta}) \}$

$= \exp [\frac{1}{2} \log 3 + \frac{1}{2} \log e^{i\theta}]$ $\quad m^n = n \log m$

$= \exp [\frac{1}{2} \log 3 + \frac{i\theta}{2}]$

$= \exp [\frac{1}{2} \log (3 + i\theta)] \quad \exp (\frac{1}{2} \log 3 + \frac{1}{2} \log i\theta)$

$= \exp \log 3^{1/2} + e^{i\theta/2} \quad = \exp (\log 3^{1/2} + \log i\theta^{1/2})$

$= 3^{1/2} \cdot e^{i\theta/2} \quad = \exp (\log 3^{1/2} \cdot e^{i\theta/2})$

$f[z(\theta)] = \sqrt{3}\, e^{i\theta/2} \quad = 3^{1/2} \cdot e^{i\theta/2}$

Hence, the right hand limits of the real and imaginary components of the function.

w.k.T, $\quad \int_C f(z)\,dz = \int_a^b f[z(t)]\, z'(t)\,dt$

$t = \theta \quad f[z(\theta)]\, z'(\theta) = \sqrt{3}\, e^{i\theta/2} (3e^{i\theta})$

$z = 3e^{i\theta}$ $\quad = \sqrt{3}\, e^{i\theta/2}\, 3ie^{i\theta} \quad e^{i\theta(1/2+1)}$

$z' = 3ie^{i\theta}$ $\quad = 3\sqrt{3}\, i\, e^{i\theta/2}\, e^{i\theta} \quad e^{i\theta(3/2)}$

$= 3\sqrt{3}\, i \cdot e^{i3\theta/2}$

$= 3\sqrt{3}\, i \left[\cos \frac{3\theta}{2} + i \sin \frac{3\theta}{2} \right]$

$= 3\sqrt{3}\, i \cos \frac{3\theta}{2} - 3\sqrt{3} \sin \frac{3\theta}{2}$

$\therefore f[z(\theta)]\, z'(\theta) = -3\sqrt{3} \sin \frac{3\theta}{2} + i 3\sqrt{3} \cos \frac{3\theta}{2} \quad (0 \le \theta \le \pi)$

At $= \theta = 0$ exist

$f[z(\theta)]\, z'(\theta) = -3\sqrt{3} \sin\theta + i 3\sqrt{3} \cos\theta$

$$= \frac{iR^a}{ia}\left[e^{ia\theta}\right]_{-\pi}^{\pi}$$

$$= \frac{2iR^a}{ia}\left[\frac{e^{ia\pi} - e^{ia(-\pi)}}{2i}\right]$$

$$= \frac{2iR^a}{a}\left[\sin a\pi\right]$$

If a is a non zero Integer n, so

$$\int_C z^{n-1}\,dz = 0 \qquad (n = \pm 1, \pm 2, \dots)$$

If a is allowed to zero, we have

i.e., $a = 0$

$$f(z) = z^{a-1}$$

$$= z^{-1}$$

$$= \frac{1}{z}$$

$$\int_C \frac{1}{z}\,dz = \int_{-\pi}^{\pi} \frac{1}{z} \cdot z'(\theta)\,d\theta$$

$$= \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} \cdot iRe^{i\theta}\,d\theta$$

$$= \int_{-\pi}^{\pi} i\,d\theta$$

$$= i\int_{-\pi}^{\pi} d\theta$$

$$= i\left[\theta\right]_{-\pi}^{\pi} \qquad = i[\pi + \pi]$$

$$= 2\pi i$$

$$\therefore \int_C \frac{dz}{z} = 2\pi i$$

3) If $f(z) = z-1$ and 'c' is the arc from $z=0$ $z=2$ consisting of the semicircle $z = 1+e^{i\theta}$ $(\pi \le \theta \le 2\pi)$ then evaluate $\int_C f(z)\,dz$.

Sol :-

Given $\int_C f(z)\,dz = \int_C (z-1)\,dz$ (OR)

$f(z) = z-1$ and c is the arc from $z=0$ to $z=2$.

Then,

$$z = 1 + e^{i\theta}$$

$$dz = ie^{i\theta}.\,d\theta$$

$$\frac{dz}{d\theta} = ie^{i\theta}.$$

Since,

$$f(z) = z^{1}-1$$

$$f[z(\theta)] = z(\theta) - 1$$

$$= 1 + e^{i\theta} - 1$$

$$f[z(\theta)] = e^{i\theta}$$

Now,

$$\int_C f(z)\,dz = \int_{\pi}^{2\pi} e^{i\theta}.\,ie^{i\theta}\,d\theta.$$

$$= i\int_{\pi}^{2\pi} e^{2i\theta}\,d\theta$$

$$= i\left[\frac{e^{2i\theta}}{2i}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left[e^{2i\theta}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left[\cos 2\theta + i\sin 2\theta\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\left[(\cos 4\pi + i\sin 4\pi) - (\cos 2\pi + i\sin 2\pi)\right]$$

$$= \frac{1}{2}\left[(1+0) - (1+0)\right]$$

$$= \frac{1}{2}(0)$$

$$= 0$$

$$\therefore \int_C f(z)\,dz = 0$$

4) If $f(z) = z-1$ and c is the arc from $z=0$ to $z=2$ consisting of the segment $z = x\ (0 \le x \le 2)$ of the real axis.

Sol:-

Given, $\int_C f(z)dz = \int_C z-1\, dz$

and c is the arc from $z=0$ to $z=2$

Then $z = x,\quad 0 \le x \le 2$

$dz = 0$

Since, $f(z) = z-1$

$f[z(0)] = x-1$

$\frac{1}{2}(2-1)^2 = 0$

$\frac{1}{2} \times 4$

$\frac{4-1}{2}$

$\frac{4}{2} - 1 = \frac{4-2}{2}$

Now,

$$\int_C f(z)dz = \int_0^2 (x-1)dx$$

$$= \left[\frac{(x-1)^2}{2}\right]_0^2$$

$$= \left(\frac{4}{2} - 2\right) - (0)$$

$$\therefore \int_C f(z)\,dz = 0$$

Upper Bounds for Moduli of contour Integrals:-

Lemma: 1

If $w(t)$ is a piece continuous complex valued function defined on an interval $a \le t \le b$

Then,

$$\left|\int_a^b w(t)dt\right| \le \int_a^b |w(t)|\,dt$$

proof:-

Let $w(t)$ is a piecewise continuous complex valued function defined on a interval $a \le t \le b$.

we prove that,

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt \rightarrow (A)$$

case (i) :

when, $\int_a^b w(t)\,dt = 0$

$$\Rightarrow \left|\int_a^b w(t)\,dt\right| = 0$$

$\therefore$ The theorem is true.

$$\therefore \left|\int_a^b w(t)\,dt\right| = 0$$

$$A \Rightarrow 0 \leq \int_a^b |w(t)|\,dt \rightarrow (1)$$

case (ii):

If $\int_a^b w(t)\,dt \neq 0$ is a complex number.

let us take,

Assume $r_0 e^{i\theta_0} = \int_a^b w(t)\,dt \rightarrow (2)$

Now,

$$r_0 = \frac{1}{e^{i\theta_0}} \int_a^b w(t)\,dt$$

$$r_0 = \int_a^b e^{-i\theta_0}\, w(t)\,dt \rightarrow (3)$$

$$|r_0| = \left|\int_a^b e^{-i\theta_0}\, w(t)\,dt\right|$$

$$\sqrt{r_0^2} = \int_a^b |e^{-i\theta_0}|\, |w(t)|\,dt.$$

$$r_0 = \int_a^b |w(t)|\,dt$$

From eqn (3),

$$r_0 = \int_a^b e^{-i\theta_0}\, w(t)\,dt$$

In this equation the Right hand side is a real part

$$\therefore r_0 = \mathrm{Re} \int_a^b e^{-i\theta_0}\, w(t)\,dt$$

$$r_0 = \int_a^b \mathrm{Re}\,[\underbrace{e^{-i\theta_0}\, w(t)}_{z}]\,dt \rightarrow (4)$$

But $\mathrm{Re}(z) \leq |z|$ formula

$$\therefore \mathrm{Re}[e^{-i\theta_0}\, w(t)] \leq |e^{i\theta_0}\, w(t)| = |w(t)|$$

$\therefore Re[e^{-i\theta_0} w(t)] \le |w(t)|$

(4) $\Rightarrow r_0 \le \int_a^b |w(t)| dt \rightarrow$ (5)

(2) $\Rightarrow r_0 e^{i\theta_0} = \int_a^b w(t) dt$

$|r_0 e^{i\theta_0}| = |\int_a^b w(t) dt|$

$|r_0| |e^{i\theta_0}| = |\int_a^b w(t) dt|$

$r_0 = |\int_a^b w(t) dt| \rightarrow$ (6)

From (5) & (6) we get $|\int_a^b w(t) dt| \le \int_a^b |w(t)| dt$

Theorem :

Let c denote a contour of length L, and suppose that a function $f(z)$ is piece wise continuous on c. If M is a non negative consta such that, $|f(z)| \le M$ for all points z on c which $f(z)$ is defined, then $|\int_c f(z) dz| \le ML$

proof :

Given c is a contour of length. So we assume the equation of c is constant.

$z = z(t)$, where $a \le t \le b$ be a parameter representation of c.

Then, M is a non-negative

ie, $|f[z(t)]| \le M \; \forall \; t; a \le t \le b \rightarrow$ (1)

Since, $z = z(t)$

$dz = z'(t) dt$

w.k.T $|\int_c f(z) dz| = |\int_a^b f[z(t)] z'(t) dt| \rightarrow$ (2)

By the lemma,

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$$

$$\Rightarrow \left|\int_a^b f[z(t)]\,z'(t)\,dt\right| \le \int_a^b |f[z(t)]\,z'(t)|\,dt$$

$$\le \int_a^b |f[z(t)]|\,|z'(t)|\,dt$$

$$\le \int_a^b M\,|z'(t)|\,dt \quad \text{using } ①$$

$$= M\int_a^b |z'(t)|\,dt$$

$$= ML \qquad z'(t) = L$$

$$\therefore \left|\int_c f(z)\,dz\right| \le ML$$

Ex:1

1) Let c be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant Such that $\left|\int_c \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}$

Sol:-

Given c be the arc of the circle $|z| = 2$.

To prove that,

$$\left|\int_c \frac{z+4}{z^3-1}\,dz\right| \le \frac{6}{7}$$

Now, consider the numerator,

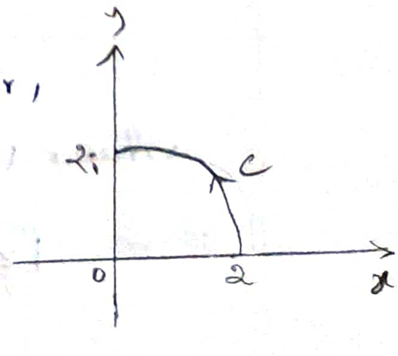
$$|z+4| \le |z| + |4|$$

$$= 2 + \sqrt{4^2}$$

$$= 2+4$$

$$|z+4| = 6 \rightarrow ①$$

Now, consider the denominator,



$$|z^3 - 1| \ge |z|^3 - 1|$$

$$= 2^3 - 1$$

$$= 8 - 1$$

$|z^3 - 1| = 7 \rightarrow (2)$

when z lies on c

$$\frac{|z+4|}{|z^3-1|} \leq \frac{6}{7}$$

let us take $M = \frac{6}{7}$

$$\therefore \left| \int_c \frac{z+4}{z^3-1} dz \right| \leq \int_c \frac{|z+4|}{|z^3-1|} dz$$

$$\leq \int_c \frac{6}{7} dz$$

$$= \frac{6}{7} \int dz$$

$$= \frac{6}{7} \pi$$

$$\therefore \left| \int_c \frac{z+4}{z^3-1} dz \right| = \frac{6}{7}\pi \rightarrow (3)$$

where π is the length of the c.

$$\therefore \left| \int_c f(z)\, dz \right| \leq ML \rightarrow (4)$$

comparing (3) and (4) we get

$M = 6/7$, $L = \pi$.

2) consider c_R is the semicircular path $z = Re^{i\theta}$ on $0 \leq \theta \leq \pi$. Then show that $\lim_{R \to \infty} \int_{c_R} \frac{z^{1/2}}{z^2+1} dz = 0$

Sol:-

Let $z = Re^{i\theta}$ on interval $0 \leq \theta \leq \pi$

By the principal of branch cut, we consider,

$$z^{1/2} = \exp\left(\frac{1}{2} \log z\right)$$

$$= \sqrt{r}\, e^{i\theta/2} \quad (r > 0, -\pi/2 < \theta < 3\pi/2)$$

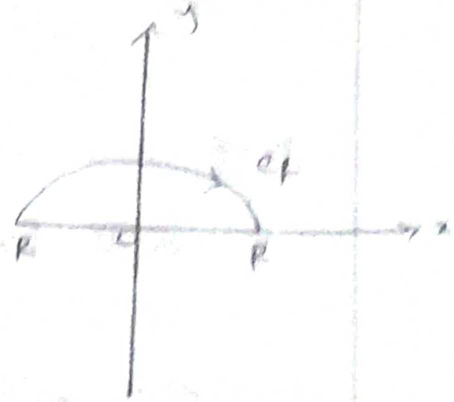
when, $|z| = R > 1$

$$|z^{1/2}| = |(Re^{i\theta})^{1/2}|$$

$= |\sqrt{R}\, e^{i\theta/2}|$

$= |\sqrt{R}|\, |e^{i\theta/2}|$

$|z^{1/2}| = \sqrt{R} \to (1)$



and,

$|z^2+1| \geq ||z^2|-1|$

$= R^2-1$

$|z^2+1| \geq R^2-1 \to (2)$

w.k.T

$|\int_C f(z)dz| \leq ML \to (3)$

$\therefore |\int_{C_R} \frac{z^{1/2}}{z^2+1} dz| \leq \int_C \frac{z^{1/2}}{z^2+1} dz$

$\leq \int_C \frac{\sqrt{R}}{R^2-1} dz$

Since, the length of C_R is the number L

$= \int_C M_R\, dz$

$= M_R \int_C dz$

$= M_R \cdot \pi R$

$= M_R \cdot L \quad | \because \pi R = L$

$\therefore |\int_{C_R} \frac{z^{1/2}}{z^2+1} dz| = M_R L$

But,

$M_R L = \frac{\sqrt{R} \cdot \pi R}{R^2-1}$

$= \frac{\pi R \sqrt{R}}{R^2-1}$

$= \frac{\pi \cancel{R} \sqrt{R}}{R^{\cancel{2}}(1-\frac{1}{R^2})}$

$= \frac{\pi \sqrt{R}}{R(1-\frac{1}{R^2})}$

$$= \frac{\pi\sqrt{R}}{\sqrt{R}\sqrt{R}\left(1-\frac{1}{R^2}\right)} = \frac{\pi}{\sqrt{R}\left(1-\frac{1}{R^2}\right)}$$

Now,

$$\lim_{R\to\infty} \left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz\right| \le \lim_{R\to\infty} \frac{\pi/\sqrt{R}}{(1-1/R^2)}$$

$$\therefore \lim_{R\to\infty} \int_C \frac{z^{1/2}}{z^2+1}\,dz = 0$$

3) Evaluateting the integral show that $\left|\int_c \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$ when c is the arc of the circle $|z|=2$ from $z=2i$.

Sol:-

Given c be the arc of the circle $|z|=2$ from $z=2$, to $z=2i$ that lies in the first quadrant. $\sqrt{z^2}=2$ $z=2$

This is done by nothing first that if z is a point on c, so that $|z|=2$, then

$$f(z) = \frac{1}{z^2-1}$$

$$|f(z)| = \left|\frac{1}{z^2-1}\right| = \frac{|1|}{|z^2-1|} \quad \frac{1}{|4-1|} = \frac{1}{|3|} = \frac{1}{3}$$

$$= \frac{1}{|z|^2-1} = \frac{1}{z^2-1} = \frac{1}{4-1}$$

$$= \frac{1}{3} \quad \because z=2$$

Thus, when z lies on c

$$|f(z)| \le M$$

$$\left|\frac{1}{z^2-1}\right| \le \frac{1}{3}$$

$M=\frac{1}{3}$ and observing that $L=\pi$ is the length of c

$$\left|\int_c \frac{1}{z^2-1}\,dz\right| \le \pi/3$$

4) Show that $\left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2}$ where c is the line segment from $z = i$ to $z = 1$.

Sol:- Let c be the circle of the consider the denominator.

$$|z^4| \geq |1|$$

$$|z^4| \geq 1$$

Let us make $M = 1$

$$\left| \int_C \frac{dz}{z^4} \right| \leq \int_C \left| \frac{1}{z^4} \right| dz$$

$$\leq \frac{1}{z^4} \int_C dz$$

$$\leq 4$$

Mid points :

$$AC^2 = AB^2 + BC^2$$

$$AC^2 = 1 + 1 = 2$$

$$AC = \sqrt{2}$$

ie., $\left| \int_C \frac{dz}{z^4} \right| \leq 4\sqrt{2}$

① Anti derivatives :- (Integration)

The concept on an anti derivative of a continuous function $f(z)$ on a domain D (or) a function $F(z)$ such that $F'(z) = f(z)$ for all z in D.

$f'(z), f''(z)$

Note :

An Antiderivatives is of necessity, an analytic function.

An Anti derivatives of a on function $f(z)$ is unique except for an additive constant.

10M

Theorem:

Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true then so are the others:

A) $f(z)$ has an antiderivative $F(z)$ throughout D.

B) The integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_2 all have the same value, namely,

$$\int_{z_1}^{z_2} f(z)\,dz = [F(z)]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (A).

c) The integrals of $f(z)$ around closed contours lying entirely in D all have value 0.

proof:

To prove this theorem it is sufficient to show that the statement $[a] \Rightarrow [b]$, $[b] \Rightarrow [c]$ then $[c] \Rightarrow [a]$.

First we prove that $[a] \Rightarrow [b]$

Suppose that $f(x)$ has an antiderivatives $F(z)$ on the domain D. To show that

$$\int_{z_1}^{z_2} f(z)\,dz = [F(z)]_{z_1}^{z_2}$$

$$= F(z_2) - F(z_1)$$

let c is contour from z_1 to z_2 is a smooth arc lying in D.

Take $z = z(t)$; $a \le t \le b$

Since we know that

$$\frac{d}{dt} F[z(t)] = F'[z(t)]\, z'(t)$$

$= F'[z(t)]\, z'(t)$; $a \le t \le b$

Now,

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\, z'(t)\,dt$$

$$= \int_a^b \frac{d}{dt} F[z(t)]\,dt$$

$$= F[z(t)]_a^b$$

$$= F[z(b)] - F[z(a)] \rightarrow (1)$$

Since,

$z(b) = z_2$ and

$z(a) = z_1$

Then (1) $\Rightarrow$

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2).F(z_1)$$

$$= F[(z)]_{z_1}^{z_2}$$

when c is any contour, not necessarily smooth one, that lies in D for it t consist of a finite number of smooth arc $c_k = 1, 2, \ldots n$) and each c_k extending from a point z_k to z_{k+1}.

$z = z_1, z_2, c_k = z_k$

$$\therefore \int_C f(z)\,dz = \sum_{k=1}^{n} \int_C c_k f(z)\,dz$$

$$= \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} f(z)\,dz \qquad [\because c_k = 1$$

$$\sum_{k=1}^{n} [F(z)]_{z_k}^{z_{k+1}}$$

$$= \sum_{k=1}^{n} F[z_{k+1}] - F[z_k]$$

$$= F[z_{n+1}] - F[z_1]$$

$$\therefore \int_C f(z)\,dz = F(z_{n+1}) - F(z_1)$$

next to prove,

$[b] \Rightarrow [c]$

Let z_1 and z_2 denoted two points on closed contour c lying in D.

To show that;

The integral around a closed contour in D in zero.

ie., $\int_C f(z)dz = 0$

Let C_1 and C_2 be two parts each part have initial point z_1 and final point z_2 show that

$C = C_1 - C_2$

Assume that the integral is independent of part in D.

$\therefore$ we can write

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

(-) $\int \rightarrow$ solution not find it

$$\Rightarrow \int_{C_1} f(z)\,dz = \left[\int_{C_2} f(z)\,dz\right] = 0$$

Limit sare interchanging so, (-) exist

$$\Rightarrow \int_{C_1} f(z)\,dz - \left[\int_{C_2}^{(-)} f(z)\,dz\right] = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

2 limit

$$\Rightarrow \int_{C_1 - C_2} f(z)\,dz = 0$$

$$\therefore \int f(z)dz = 0 \rightarrow (2)$$

ie., $f(z)$ is around the closed contour $C = C_1 - C_2$ has the value zero.

ie $f(z)$ is around the closed contour.

Next to show that,

$[c] \Rightarrow [a]$

To show that, $f(x)$ has an antiderivatives $F(x)$ $F(z)$

Let as assume that,

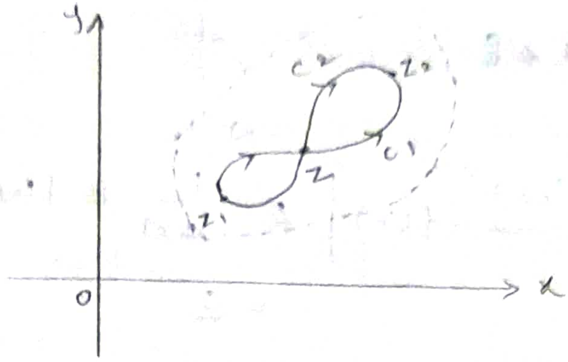
$$\int_C f(z)\,dz = 0$$

Let C_1 and C_2 are two contours such that,

$C = C_1 - C_2$ lying in D from a point z_1 to a point z_2.

ie $\int_{C_1 - C_2} f(z)\,dz = 0$

$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$



Define,

$$F(z) = \int_{z_0}^{z} f(s)\,ds \text{ on } D \rightarrow (3)$$

It is enough to show that

$$F'(z) = f(z)$$

for complete proof of the theorem

let $z+\Delta z$ be the distinct point from z lying in some neighbourhood of z.

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds$$

$\div \Delta z$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\int_{z_0}^{z+\Delta z} f(s)\,ds - \frac{1}{\Delta z}\int_{z_0}^{z} f(s)\,dz$$

(-) exist limit into

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(s)\,ds + \int_{z}^{z_0} f(s)\,ds\right]$$

$$\frac{F(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z}^{z+\Delta z} f(s)\,ds\right] \rightarrow (4)$$

But f is continuous at the point z.

Hence for each positive number ε a positive number δ exists.

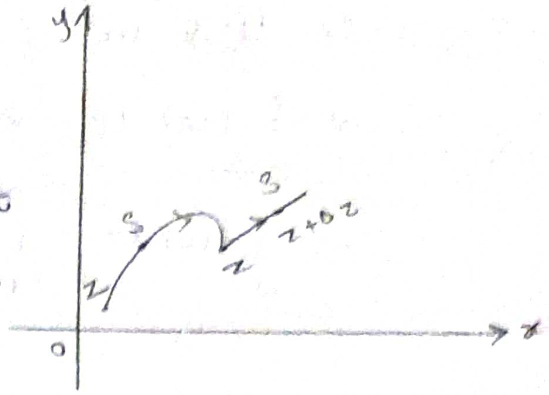
such that,

$$|f(s) - f(z)| < \varepsilon$$

whenever $|s-z| < \delta$

consequently, if the point $z+\Delta z$ is close enough to z.

so that $|\Delta z| < \delta$



Then,

$$\left| \frac{f(z+\Delta z) - f(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \varepsilon \, |\Delta z|$$

$$= \varepsilon$$

That is,

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = F(z)$$

(OR)

$$\therefore F'(z) = f(z)$$

$\therefore f(z)$ the has an 'Antiderivative of $F(z)$.'

problems :-

1) Show that the function $f(z) = \frac{1}{z^2}$ which is continuous every where except at the orgin $F(z) = -\frac{1}{z}$ is $|z| > 0$. when $z = 2e^{i\theta}$ $(-\pi \le \theta \le \pi)$.

Sol:-

The given function $f(z) = \frac{1}{z^2}$ is continuous every where except at 0.

To show that $\frac{1}{z^2}$ has an derivative $-\frac{1}{z}$

Given, $z = 2e^{i\theta}$

$$dz = 2ie^{i\theta} \cdot d\theta \; ; \; -\pi \le \theta \le \pi$$

Now,

$$\int_C f(z)dz = \int_C \frac{1}{z^2} dz$$

where c is the circle $|z| = 2$

$$\therefore \int_C \frac{1}{z^2} dz = \int_{-\pi}^{\pi} \frac{1}{(2e^{i\theta})^2} \cdot 2ie^{i\theta} d\theta$$

$$= \int_{-\pi}^{\pi} \frac{1}{4(e^{i\theta})^2} 2ie^{i\theta}\, d\theta.$$

$$= \frac{i}{2} \int_{-\pi}^{\pi} e^{-i\theta} d\theta$$

$$= \frac{i}{2} \left[\frac{e^{-i\theta}}{-i}\right]_{-\pi}^{\pi}$$

$$= \frac{i}{2} \left[\frac{\cos\theta - i\sin\theta}{-i}\right]_{-\pi}^{\pi}$$

$$= \frac{-1}{2}\left[(\cos\pi - i\sin\pi) - (\cos(-\pi) - i\sin(-\pi))\right]$$

$$= \frac{-1}{2}[-1-0+1-0]$$

$$= 0$$

$$\therefore \int_C \frac{1}{z^2} dz = 0 \rightarrow (1)$$

Next, $F(z) = \frac{-1}{z}$; $|z|>0$

when, $z = e^{i\theta}$; $(-\pi \le \theta \le \pi)$

$$[F(z)]_{-\pi}^{\pi} \Rightarrow \left[\frac{-1}{e^{i\theta}}\right]_{-\pi}^{\pi}$$

$$\Rightarrow [-e^{-i\theta}]_{-\pi}^{\pi}$$

$$F(z) = 0 \rightarrow (2)$$

From (1) & (2)

$$\Rightarrow \int_C f(z)dz = F(z) \Rightarrow f(z) = F'(z)$$

$$\therefore \int_C \frac{1}{z^2} dz = \frac{-1}{z}$$

2) Show that the continuous function $f(z) = z^2$ has anti derivative $F(z) = \frac{z^3}{3}$ show throughout the plane for every contour $z=0$ to $z=1+i$.

5m

Sol:-

The gn function $f(z) = z^2$ which continuous through out the plane.

To show that $F(z) = \frac{z^3}{3}$ is the anti derivative $f(z)=z^2$.

$$\int_C f(z)\,dz = \int_C z^2\,dz$$

where c is the 0 to $1+i$

$$= \int_0^{1+i} z^2\,dz$$

$$\int_C f(z)\,dz = \left[\frac{z^3}{3}\right]_0^{1+i}$$

$$= \frac{1}{3}\left[z^3\right]_0^{1+i}$$

$$= \frac{1}{3}\left[(1+i)^3 - 0\right]$$

$$= \frac{1}{3}\left[1+i^3+3i+3i(i)^2\right]$$

$$= \frac{1}{3}\left[1-i+3i-3\right]$$

$$= \frac{1}{3}\left[-2+2i\right]$$

$$\int_C f(z)\,dz = -\frac{2}{3}\left[1-i\right] \rightarrow (1)$$

$$F(z) = \frac{z^3}{3} \quad ; \quad z=0 \text{ to } z=1+i$$

$$\left[F(z)\right]_0^{1+i} = \left[\frac{z^3}{3}\right]_0^{1+i}$$

$$= \frac{1}{3}\left[(1+i)^3 - 0\right]$$

$$= \frac{1}{3}\left[-2+2i\right]$$

$$\left[F(z)\right] = -\frac{2}{3}\left[1-i\right] \rightarrow (2)$$

From (1) & (2) $\Rightarrow \int_C f(z)dz = F(z) \Rightarrow F(z) = F'(z)$

ie., $\int_C z^2\,dz = \frac{z^3}{3}$

3) Show that $f(z) = \frac{1}{z}$ has two different antiderivatives around C, where c is the positively oriented circle

$z = 2e^{i\theta}$, $-\pi/2 \le \theta \le \pi/2$

Sol:- Given $f(z) = \frac{1}{z}$

consider the priciple branch,

$z = re^{i\theta}$, $r>0$, $-\pi/2 \le \theta \le \pi/2$

$\log z = \log(re^{i\theta})$ $\qquad \log mn = \log m + \log n$

$= \log r + \log e^{i\theta}$ $\qquad \log m^n = n \log m$

$\log z = \log(r + i\theta) \rightarrow ①$

when' $z = 2e^{i\theta}$, $(-\pi/2 < \theta < \pi/2)$

$\Rightarrow \log z = \log z + i\theta$

$$\int_{C_1} \frac{1}{z}\, dz = \int_{-2i}^{2i} \frac{1}{z}\, dz \qquad \because C = C_1 + C_2$$

where c_1 is the contour right half the integral of $\frac{1}{z}$

$$= [\log z]_{-2i}^{2i},$$

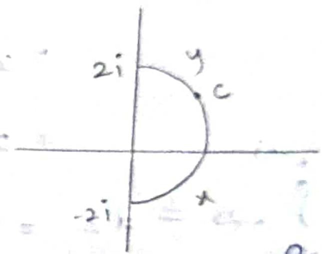
$$= \log 2i - \log(-2i)$$

$$= \log 2 + \log i - [\log 2 + \log(-i)]$$

$$= \cancel{\log 2} + \log e^{i\pi/2} - \log 2 - \log e^{-i\pi/2} \qquad \theta = -\pi/2, \pi/2$$

$$= i\pi/2 - (-i\pi/2)$$

$$= i\, \cancel{2}\pi/\cancel{2}$$



$$\int_{-2i}^{2i} \frac{1}{z}\, dz = \pi i \rightarrow ②$$

where c_2 is left half of the integral of $\frac{1}{z}$.

next $\int_{C_2} f(z)dz = \int_{2i}^{-2i} \frac{1}{z}\, dz$

$$= [\log z]_{2i}^{-2i}$$

$$= \log(-2i) - \log(2i)$$

$$= \cancel{\log 2} + \log(-i) - \log 2 - \log i$$

$$= e^{-i3\pi/2} - i\pi/2$$

$$= i\left[\frac{3\pi}{2} - \pi/2\right]$$

$$= i\,[\cancel{2}\pi/\cancel{2}]$$

$$\therefore \int_{C_2} f(z)\, dz = \pi i \rightarrow ③$$

From ③ & ④

$$\int_C \frac{1}{z}\,dz = \int_{C1} \frac{1}{z}\,dz + \int_{C2} \frac{1}{z}\,dz$$

$$= \pi i + \pi i$$

$$\int_C \frac{1}{z}\,dz = 2\pi i$$

④ Evaluate $\int_0^{\pi+2i} \cos z/2 \, dz$

Sol:-

Given $f(z) = \cos \frac{z}{2}$ from 0 to $\pi+2i$

$$\int_C f(z)dz = \int_0^{\pi+2i} \cos \frac{z}{2}\,dz$$

$$= \left[\frac{\sin z/2}{1/2}\right]_0^{\pi+2i} \qquad \frac{1}{2}\sin z$$

$$= 2\left[\sin\frac{(\pi+2i)}{2} - \sin 0/2\right]$$

$$\sin \pi/2 + \frac{2i}{2}$$

$$= 2\sin(\pi/2 + i)$$

$$= 2(\sin \pi/2 \cos i + \cos \pi/2 \sin i)$$

$$= 2(\cos i)$$

$$\int_0^{\pi+2i} \cos \frac{z}{2}\,dz = 2\cosh \rightarrow ①$$

We know that,

$$e^{\theta} = \left(1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \cdots\right) + \left(1 - \frac{\theta}{1!} + \frac{\theta^2}{2!}\right)$$

$$e^{-\theta} = 1 - \frac{\theta}{1!} + \frac{\theta^2}{2!} + \cdots$$

$$e^{\theta} + e^{-\theta} = 2 + \frac{2\theta^2}{2!} + \frac{2\theta^4}{4!} + \cdots$$

$$= \left[1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right]$$

$$= 2\cosh\theta$$

where $\theta = 1$

$$e + \frac{1}{e} = 2\cosh$$

$$① \Rightarrow \int_0^{\pi+2i} \cos \frac{z}{2}\,dz = e + \frac{1}{e}$$

the anti derivatives to show that for contours

C extending from a point z_1 to

$$\int_C z^n dz = \frac{1}{n+1}\left[z_2^{n+1} - z_1^{n+1}\right].$$

Sol:- The gn function $f(z) = z^n$

To prove that, $\int_C z^n dz = \frac{1}{n+1}\left[z_2^{n+1} - z_1^{n+1}\right]$

Now, $\int_C z^n dz = \int_{z_1}^{z_2} z^n dz$

$$= \left[\frac{z^{n+1}}{n+1}\right]_{z_1}^{z_2}$$

$$= \frac{1}{n+1}\left[z^{n+1}\right]_{z_1}^{z_2}$$

$$\int_C z^n dz = \frac{1}{n+1}\left[z_2^{n+1} - z_1^{n+1}\right] \to ①$$

Next, $F(z) = \left[\frac{z^{n+1}}{n+1}\right]_{z_1}^{z_2}$

$$= \frac{1}{n+1}\left[z^{n+1}\right]_{z_1}^{z_2}$$

$$F(z) = \frac{1}{n+1}\left[z_2^{n+1} - z_1^{n+1}\right] \to ②$$

$$\int_C f(z)\,dz = F(z)$$

$$f(z) = F'(z)$$

ie, $\int_C z^n dz = \frac{1}{n+1}\left[z_2^{n+1} - z_1^{n+1}\right]$

$\therefore \int_C z^n dz$ has an antiderivatives $\frac{z^{n+1}}{n+1}$

b) Evaluate $\int_i^{i/2} e^{\pi z} dz$ w.r.t. πz

Sol:-

$$f(z) = e^{\pi z}$$

Now, $\int_C f(z)dz = \int_i^{i/2} e^{\pi z} dz$

$$= \left[\frac{e^{\pi z}}{\pi}\right]_{i}^{i/2}$$

$$= \frac{1}{\pi}\left[e^{\pi z}\right]_{i}^{i/2}$$

$$= \frac{1}{\pi}\left[e^{i\pi/2} - e^{i\pi}\right]$$

$$= \frac{1}{\pi}\left[(\cos \pi/2 + i\sin \pi/2) - (\cos\pi + i\sin\pi)\right]$$

$$= \frac{1}{\pi}\left[(0+i) - (-1+0)\right]$$

$$= \frac{1}{\pi}\left[i+1\right]$$

$$= \frac{1}{\pi}\left[1+i\right]$$

$$\int_{i}^{i/2} e^{\pi z}\,dz = \frac{1+i}{\pi} \rightarrow ①$$

$$F(z) = \left[\frac{e^{\pi z}}{\pi}\right]_{i}^{i/2}$$

$$= \frac{1}{\pi}\left[e^{\pi z}\right]_{i}^{i/2}$$

$$= \frac{1}{\pi}\left[(\cos \pi/2 + i\sin\pi/2) - (\cos\pi + i\sin\pi)\right]$$

$$= \frac{1}{\pi}\left[(0+i) - (-1+0)\right]$$

$$= \frac{1}{\pi}\left[1+i\right]$$

$$F(z) = \frac{1+i}{\pi} \rightarrow ②$$

From ① & ②

$\therefore \int_{i}^{i/2} e^{\pi z}\,dz$ has an anti derivatives $\frac{1+i}{\pi}$.

Cauchy - theorem (or) Cauchy - Goursat Theorem:-

5M

Statement :- 2M

Let f be an analytic function at each point interior to and on C.

Then,

$$\int_{C} f(z)\,dz = -\int_{-C} f(z)\,dz$$

proof :

let f be an analytic function at each point on C. where C denote simple closed contour

$z(t)$; $(a \le t \le b)$ $\to z(a)$

To show that,

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$$

By W.R.T,

formula $\int_C f(z)\,dz = \int_C f[z(t)]\; z'(t)\,dt \to ①$ AR

If, $f(z) = u(x,y) + iv(x,y)$ and

$z(t) = x(t) + iy(t)$ and

$z'(t) = x'(t) + iy'(t)$

$\therefore f[z(t)]\,z'(t) = \{u[x(t), y(t)] + iv[x(t), y(t)]\}\cdot[x'(t) + iy'(t)]$

$= u x' + iv x' + iuy' - vy'$

$= ux' - vy' + i(vx' + uy')$

$① \Rightarrow \int_C f(z)dz = \int_a^b [(ux' - vy') + i(vx' + uy')]\,dt$

$= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt$

$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy \to ③$

we know that,

By the Green's theorem,

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA \to ④$$

use ④ in ③ we get

$$\int_C f(z)\,dz = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA \to ⑤$$

By the C-R equation,

By the C-R equation

$u_x = v_y$

$u_y = -v_x$

(5) $\Rightarrow \int_C f(z)\,dz = \iint_R (u_y - u_y)\,dA + \iint_R (v_y - v_y)\,dA$

$\therefore \int_C f(z)\,dz = 0 \rightarrow$ (6)

Since Eq (6) true when c taken contour clockwise direction. (-)

Also (6) is true c taken clock wise (+) direction.

$\therefore -\int_{-C} f(z)\,dz = 0$

$\therefore \int_C f(z)\,dz = -\int_{-C} f(z)\,dz$

(OR)

$\int_C f(z)\,dz = 0$ H/P

Cauchy - Goursat theorem :-

Statement :-

10m (1) If a function f is analytic at all points interior to and on a simple closed contour c then $\int_C f(z)\,dz = 0$.

proof :

To prove this theorem, we need lemma as

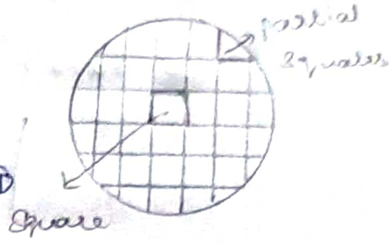
2M Goursat lamma.

Statement of lemma :- Goursat lemma

let f be analytic function on a closed

2M region R consisting of the points interior to a positively oriented simple closed contour c together with the points on c itself.

can be concered with a finite numbers of sequences and partial squares indexed by $j = 1, 2, \dots n$ such that in each one there is a fixed point z_j.



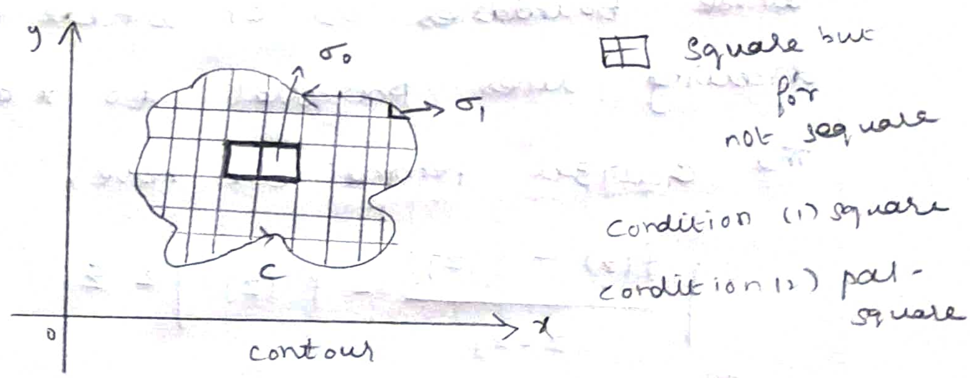
$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon \rightarrow (1)$$

is satisfied by all points other than z_j is the square or partial square.

proof of the lemma:

Let us assume that the lemma is false. ie, It is not true atleast in one square. Let us subdivide the squares by joining the middle point of the opposite sides.

In case, there is still remain say one part of is which the conditions (1) is not true.



we shall again sub divide them in the sequence way.

The above process may and either after a finite number of steps when condition (1) is true for every subdivision.

In the second case we obtained a sequence of squares which has z_j as its limit points of which the condition ① is not true.

Hence at point z_j

we have,

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon$$

where $|z - z_j|$ is small.

$\Rightarrow f(z)$ is not differentiable at $z = z_j$

i.e., $f(z)$ is not analytic at z_j

This is contradiction to the hypothesis that $f(z)$ is analytic all the points with the and on c. The lemma is true.

proof of the main theorem:

Take the region bounded by the curve c we divided into squares and partial squares whose boundaries are c_r where $r = 1, 2, \ldots, n$ by drawing lines parallel to x and y axis hence the Goursat lemma we have,

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon$$

when $\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \rightarrow ②$

at $z \neq z_j$

① $\Rightarrow |\delta_j(z)| < \varepsilon \rightarrow ③$

Now ② $\Rightarrow \delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - \frac{(z - z_j) f'(z_j)}{z - z_j}$

$\Rightarrow \delta_j(z)(z-z_j) = f(z) - f(z_j) - (z-z_j) f'(z_j)$

$\Rightarrow f(z) = f(z_j) + (z-z_j) f'(z_j) + \delta_j(z)(z-z_j) \to (4)$

suppose that the integral has been taken in contour clockwise sense around each of them

$$\int_C f(z)dz = \sum_{r=1}^{n} \int_{C_r} f(z)dz \to (B)$$

sub (4) in (B) we get,

$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{C_r} \{ f(z_j) + (z-z_j) f'(z_j) + (z-z_j)\delta_j(z) \} dz$$

$$= \sum_{r=1}^{n} \int_{C_r} \{ f(z_j) + z f'(z_j) - z_j f'(z_j) + (z-z_j)\delta_j(z) \} dz$$

$$= \sum_{r=1}^{n} \{ [f(z_j) - z_j f'(z_j)] \int_{C_r} dz + f'(z_j) \int_{C_r} z\,dz + \int_{C_r} (z-z_j)\delta_j(z)\,dz \} \to (5)$$

Now $\int_{C_r} dz = 0$ and $\int_{C_r} z\,dz = 0$

when C_r is a closed curve.

$$(5) \Rightarrow \int_C f(z)dz = \sum_{r=1}^{n} \int_{C_r} (z-z_j)\,\delta_j(z)\,dz$$

$$\Rightarrow \left| \int_C f(z)dz \right| \leq \Sigma \left| \int_{C_r} (z-z_j)\,\delta_j(z)\,dz \right|$$

$$\leq \Sigma \int_{C_r} |z-z_j|\,|\delta_j(z)|\,dz$$

$$< \varepsilon \sum_{r=1}^{n} \int_{C_r} |z-z_j|\,dz \to (A)$$

$\because |\delta_j(z)| < \varepsilon$

Clearly each boundary C_r consider with a complete (or) part of square of it and let a_r be the length of the side of square.

Also the diagonal of the square in $\sqrt{2}\, a_r$.

Now the point z is on C_r and z_j may be either on the boundary or with in the square

such as

$|z - z_j| \leq \sqrt{2}\, a_r$

$\therefore \int_C |z - z_j|\, dz \leq \sqrt{2}\, a_r \int_{C_r} dz$.

Since C_r is the complete square then $\int_{C_r} dz = 4a_r$

If it is partial square if can't exceed $(4a_r + L_r)$, where L_r is the length r the arc of the contour C with constitutes the part C_r.

If C_r is a square and A_r is its area then $\int_C |z - z_j|\, dz \leq \sqrt{2}\, a_r^2$

$= 4\sqrt{2}\, A_r \rightarrow (6)$

If it is a partial square $\quad | \because a_r^2 = A_r$

$\int_C |z - z_j|\, dz \leq \sqrt{2}\, a_r (4a_r + L_r)$

$= 4\sqrt{2}\, A_r + \sqrt{2}\, S L_r \rightarrow (7)$

From (A), (6) and (7) we get,

$\left| \int_{C_r} f(z)\, dz \right| \leq \varepsilon \left(4\sqrt{2}\, S^2 + \sqrt{2}\, SL \right)$

$= \varepsilon B \rightarrow (8)$

where B is constant.

ε is arbitrary small

$\left| \int_{C_r} f(z)\, dz \right| = 0$

$\Rightarrow \left| \int_C f(z)\, dz \right| = 0$

$\therefore \int_C f(z)\, dz = 0$.

Cauchy goursat theorem

Eg

$\int_C \frac{dz}{z+4}$ where $c : z$

$\int_C e^{z} dz$, where $c : |z|$

$\int_C e^{z^3} dz$, where $c : |z|$

Simple connected Domains : [SCD]

A simply connected Domain D is domain such that every simple closed contour with in in enclosed only points of D.

Ex:-

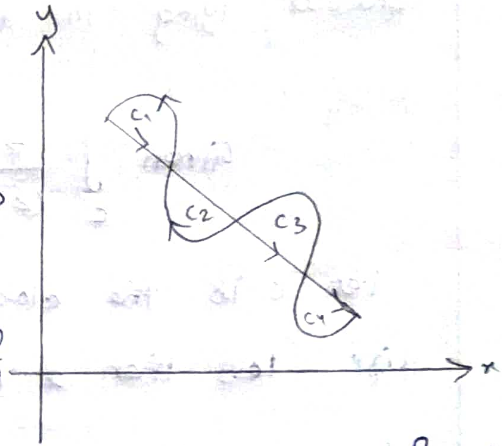
The set of points interior to a simple closed contour.

Theorem:

If a function small f is analytic through out simply connected domain D then $\int_C f(z)\,dz = 0$ for every contour c lying in D.

proof:

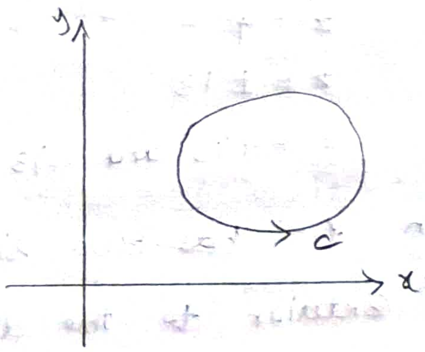
let 'c' is a simple closed contour (or) c is closed contour that intersects itself a finite number of times.



case (i)

If c is a simple closed curve and lies in D and the function $f(z)$ is analytic at each point of interior to and on c.

∴ By the cauchy Goursat theorem, a function f is analytic at all points interior to and on a simple closed contour c. Then $\int_C f(z)\,dz = 0$



case (ii)

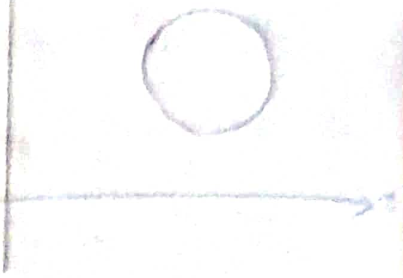
If c is closed but intersects itself a finite number of times it consists of a finite number of simple closed contours

$C_k = (k = 0, 1, 2, \ldots)$

Since the value of integral around each C_k is zero

$$\int_C f(z)\,dz - \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

$(k = 1, 2, \dots, n)$



ie, $$\int_C f(z)\,dz - \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

||| p

1) Evaluate $\int_C \frac{ze^2}{(z^2+9)^5}\,dz = 0$, where c is the closed contour lying in the opend disk < 2.

proof :

Given $$\int_C \frac{ze^2}{(z^2+9)^5}\,dz = 0$$

let c is the closed contour lying in the open disk less then 2 (<2).

Now, consider the pole,

$$(z^2+9)^5 = 0$$

$$z^2+9 = 0$$

$$z^2 = 9$$

$$z = \pm i3$$

$$z = +i3 \text{ and } -i3$$

$\therefore$ The function $f(z)$ has two singular points $z = \pm 3i$ are 'Exterior' to the disk.

$$\int_C \frac{ze^2}{(z^2+9)^5}\,dz = 0$$

Corollary :

'A function f that is analytic throughout the simply connected domain D must has an antiderivative everywhere in D.

Multiple connected domain : (MCD)

A Domain that is not simply connected is said to be multiply connected.

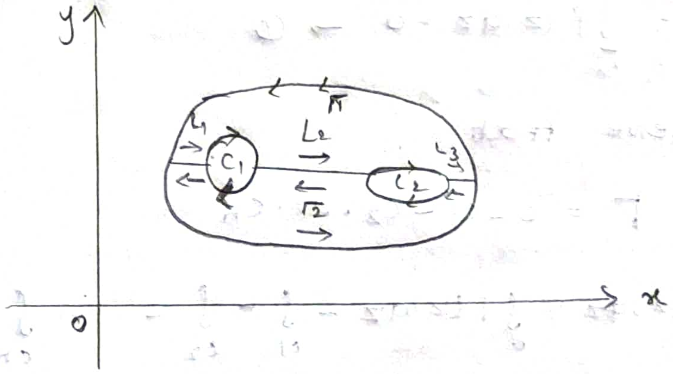
Theorem :

suppose that

a) c is the simple closed contour described in the counter (+) clock wise direction.

b) $C_k = (k = 1, 2, \dots)$ are simple closed contours interior to C, all described in the clock wise direction, that are disjoint and (−) whose interiors have no points in common. If a function f is analytic and all of this contours and throughout the multiply connected domain consisting of the points inside c and exterior to each C_k, $\int_c f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$.

proof :



Let L_1 be the polygonal path consisting of a finite number of line segments joined end to end, to connect the outer contour c to the inner contour C_1.

we introduce another polygonal path L_2 which connects C_1 to C_2 and we continue in this manner with L_{n+1} connecting C_n to C. As indicated by the single barbed arrows.

Two simple closed contours Γ_1 and Γ_2 and be formed each consisting of polygonal paths

L_K or L_K and pieces of C and C_K and each described in a such a direction that the points enclosed by them lie to the left.

Now we applied the cauchy - goursat theorem to f on Γ_1 and Γ_2, and the sum of the values of the integrals over those contours is found to be zero. Since, the integrals in opposite directions along each path L_K cancel, only the integrals along C and the C_K.

$$\Rightarrow \int_{\Gamma_1} f(z)dz = 0 \text{ and } \int_{\Gamma_2} f(z)dz = 0$$

Also,

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = \int_{\Gamma} f(z)dz$$

Since the integrals along L_j are taken finite in opposite directions

$$\therefore \int_{\Gamma} f(z)dz = 0 \rightarrow (1)$$

we observe that

$$\Gamma = C - C_1 - C_2 \cdots C_n$$

$$\text{ie.,} \int f(z)dz = \int_C F(z)dz - \int_{C_1} - \int_{C_2} - \cdots \int_{C_n}$$

$$= \int_C - \sum_{K=1}^{n} \int_{C_K}$$

$$= \int_C \sum_{K=1}^{n} \int_{-C_K} f(z)dz$$

when each C_K are transvered in the positive sence we get,

$$\int_{\Gamma} f(z)dz = \int_C f(z)dz - \int_{C_1} - \int_{C_2} \cdots - \int_{C_n} + \sum_{K=1}^{n} \int_{C_K} f(z)dz$$

By using (1) we get,

$$\int_C f(z)dz + \sum_{K=1}^{n} f(z)dz = 0$$

Since the integrals in opposite direction along each path C_K cancel, only the integrals along C and C_K.

corollary:

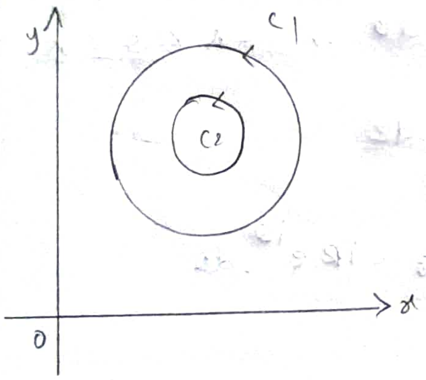
Extension of cauchy integral theorem

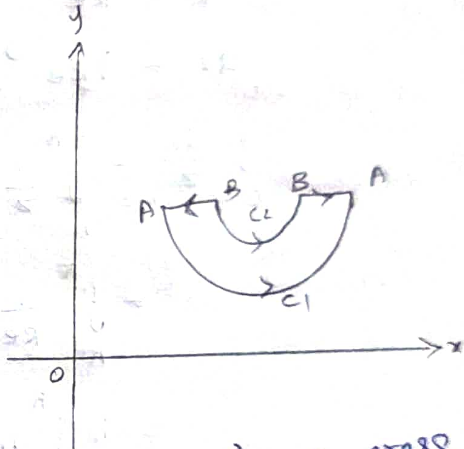
Statement:

Let C_1 and C_2 denote positively oriented simple closed contours where C_1 is interior to C_2 if a function f is analytic in the closed region consisting of those contours and all points between them then

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz$$

proof:





From the diagram by introducing a cross AB, the annual region is converted in the region bounded into single curve.

Now apply the cauchy theorem, to the contours C_1 AB, C_2 BA so that

$$\int_{C_1 AB\, C_2 BA} f(z)\,dz = 0$$

$$\therefore \int_{C_1} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \cancel{\int_{AB} f(z)\,dz} + \int_{C_2} f(z)\,dz - \cancel{\int_{AB} f(z)\,dz} = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz - \int_{-C_2} f(z)\,dz = 0$$

where C_1 and C_2 are transvesed in

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Show that $\int_C \frac{1}{z}\,dz = 2\pi i$, where c is the positively oriented circle.

Sol:-

Given $f(z) = \frac{1}{z}$

let us take,

$$z = Re^{i\theta}$$

$$dz = iRe^{i\theta}.\,d\theta \quad , \quad -\pi \le \theta \le \pi$$

$$\int_C \frac{dz}{z} = \int_{-\pi}^{\pi} \frac{1}{z}.\,dz$$

$$= \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} \cdot iRe^{i\theta}.\,d\theta$$

$$= i\int_{-\pi}^{\pi} d\theta$$

$$= i\,[\theta]_{-\pi}^{\pi}$$

$$= i[\pi + \pi]$$

$$\int_C \frac{dz}{z} = 2\pi i$$