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COMPLEX ANALYSIS

B.Sc. MATHEMATICS

$\overline{V}$ - SEMESTER

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complex Analysis

UNIT - I

Region in the complex plane - functions of a complex variable - limits - theorem on limits - limits involving at the point at infinity - continuity - derivative riemann equations 1 - sufficient condition for differentiability - polar co-ordinates - Analytic functions - examples - harmonic functions

chap 1 : (sec 11 only) chap 2: (sec 12, 15, 16 to 26).

UNIT - II

derivative of functions w(t) - definite integrals of functions w(t) - contours - contour integrals - some Examples - Examples with branch cuts - upper bounds for moduli of contour integrals - anti derivatives - proof of theorem - cauchy Goursat theorem - proof of theorem - simply connected domains - multiply connected domains.

chap 4 : sec (37 to 49).

UNIT - III

cauchy integral formula - an extension of cauchy integral formula - some consequences of extension - Liouville's theorem and the fundamental theorem of algebra - maximum modulus principle

chap 4 : (sec 50 to 54)

UNIT - IV

Mappings - Mappings by exponential function. linear transformations - the transformation - an implicit form.

chap 2 : (sec 13, 14) chap 8 (sec 19 to 92)

UNIT - V

The transformation w = sin z, w = cos z, w = sinh z, w = cosh z, Mappings by z^2 and branches of $z^{1/2}$ - conformal mappings preservation of Angles - Scale factors and local inverse.

chap 8 : (sec 96 to 97) chap 9 : (sec 101 to 103)

Text book :

James ward Brown & Ruel V. churchill I, complex variables and its application Mc Grawhill Inc 8th edition.

Reference book :

1. Gupta, Kedarnath & Ramnath, complex variables, Meerut delhi.

2. I. N. sharma functions of a complex variables krishaprakashan Media (P) Ltd. 13th edition 1996 - 97.

3. T. K. Manickavasagam pillai complex analysis - S. viswanathan publishes pvt. Ltd.

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UNIT - I

Region in the complex plane - functions of a complex variable - limits - theorem on limits - limits involving at the point at infinity - continuity - derivative riemann equations 1 - sufficient condition for differentiability - polar co-ordinates - Analytic functions - Examples - harmonic functions

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UNIT - I

Complex Numbers:

There is no real number which satisfies the equation

$$x^2+1=0$$

To permit the solution

ie) $x^2+1=0$

$x^2=-1$

$x=\pm i$

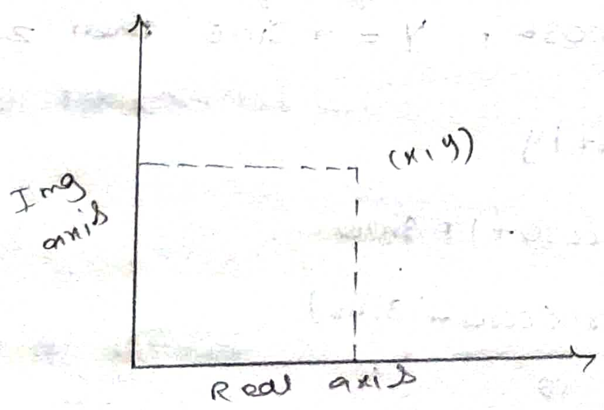
A complex number z is of the form $x+iy$ where x & y are real numbers and i is an Imaginary unit. here x and y are called the real and Imaginary parts of z

ie) $x = Re\ z$ and $y = Img\ z$

A complex number is denoted by C

ie) $C = \{x+iy \mid x, y \in R\}$

A complex number is also be defined in ordered pairs (x, y) of real numbers that are to be interpreted as a points in a complex plane with rectangular co-ordinates x & y



Eg:-

$z = 2+3i$ (ie) $(2, 3)$

Results / properties :-

1) $|z| = \sqrt{x^2+y^2}$

2) $\bar{z} = x - iy$ (complex conjugate)

3) $\bar{\bar{z}} = z$ for all z

4) $\mathrm{Re}\, z = \frac{z+z}{2}$ and $\mathrm{Img}\, z = \frac{z-z}{2i}$

5) $\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$

6) $z - \bar{z} = |z|^2$

7) $\overline{z_1 z_2} = \bar{z_1}\,\bar{z_2}$

8) $\left(\frac{\bar{z_1}}{z_2}\right) = \frac{\bar{z_1}}{z_2}$ for $z_2 \neq 0$

9) $|z| \geq 0$ and $|z| = 0$ If $z=0$

10) $|z_1 z_2| = |z_1||z_2|$

11) $|z_1+z_2| \leq |z_1|+|z_2|$

12) $|z_1-z_2| \geq |z_1|-|z_2|$

13) $|z_1+z_2|^2 \leq |z_1|^2+|z_2|^2$ ~~comp~~ (Triangle Inequality)

Complex number in polar co-ordinates :-

Let (r, θ) be a polar co-ordinates of z.

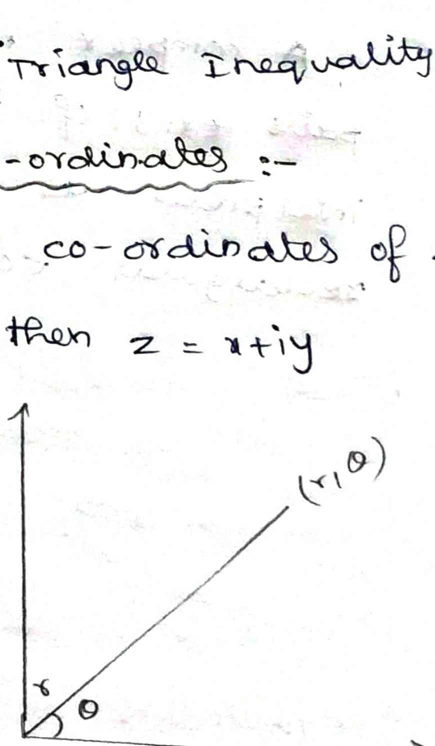
Let $x = r\cos\theta$, $y = r\sin\theta$ then $z = x+iy$

$z = x+iy$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$



where r and θ is magnitude and amplitude / argument of z.

Regions in the complex plane:

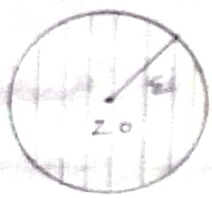
Here, we are concerned with the set of complex Numbers (or) point in z plane and their closeness to one another.

Neighborhood of z_0:

It consists of set of all points of z lying inside but not on the circle centered at z_0 and with a specified positive radius ε

ie) $|z - z_0| < \varepsilon$

The neighborhood of z_0 is denoted by $N(z_0)$ (or) $N(z_0, \varepsilon)$



Deleted neighborhood:

It consists of all points of z in an ε - neighbourhood of z_0 except for the point z_0 itself.

ie) $0 < |z - z_0| < \varepsilon$

It is also called punctured Disk.

Interior point:

A point z_0 is said to be an interior point of a set "S" whenever there is some neighborhood of z_0 that contains only a point of S

Exterior point :

A point z_0 is said to be an exterior point of S when there exists a neighborhood of z_0 that contains no points of S.

Boundary point :

A Boundary point is a point of all whose neighborhood contains atleast one point in S and atleast one point not in S. The totality of all boundary points is called the boundary of S.

Eg:-

The circle $|z| = 1$, the boundary of each of sets $|z| < 1$ and $|z| \leq 1$.

Open set :

A set S is open if it contains none of its boundary points.

ie) A set S is open iff each of its point is an interior point.

Eg:-

$|z| < 1$

Closed set :

A set is ~~called~~ ~~is~~ closed if it contains all of its boundary point

closure :

A closure of a set S is a closed set consisting of all points in S together with the boundary of S.

Eg:-

$|z| \leq 1$

Connected set :

An open set S is connected if each pair z_1 and z_2 in it can be joined by a polygonal line consisting of a finite number, of line segments joined end to end, that lies entirely in S.

Eg:-

* $|z| < 1$

* The annulus $1 < |z| < 2$ is also connected.

Domain and Region :

A non empty open set that is connected is called a domain. A domain together with some, none or all of its boundary points is referred to as a region.

Note :

Any neighborhood is a domain

Bounded :

A set S is bounded if every points of S lies inside some circle $|z| = R$, otherwise it is unbounded.

Eg:-

$|z| \leq 1$ * $\rightarrow$ Bounded

$\operatorname{Re} z \geq 0$ $\rightarrow$ unbounded

Accumulation point / cluster point / Limit point:

A point z_0 is said to be an accumulation point of a set S if each deleted neighborhood of z_0 contains atleast one point of S.

A point z_0 is not an accumulation point of S whenever there exists some deleted neighborhood of z_0 that does not contain atleast one point of S;

NOTE:

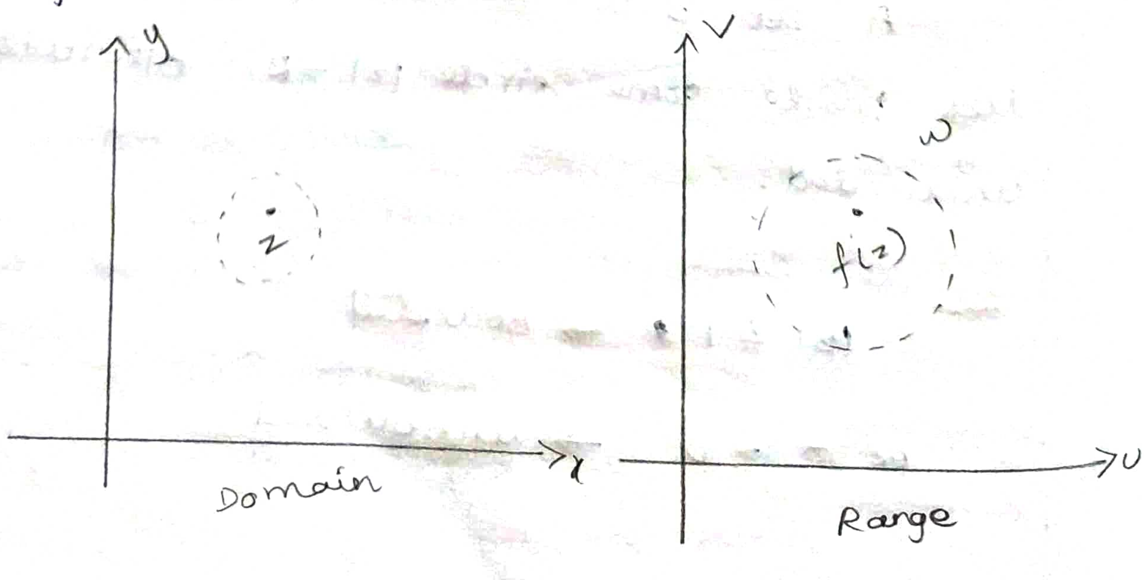
origin is the only accumulation point of the set $z_n = i/n \ (n = 1, 2, 3, \cdots \infty)$

Functions of A complex variable:

Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each z in S a complex number w. The no. w is called the value of f at z and is denoted by $f(z)$

ie) $w = f(z)$

The set S is called the domain of definition of f.



Examples :-

① If f is defined on the set $z \neq 0$ by the means of eqn $w = \frac{1}{z}$, it may be referred to only as a function $w = 1/z$ (or) simply a function $1/z$.

② we know that $w = f(z)$

suppose that $w = u+iv$ is the function f at $z = x+iy$, so that

$u+iv = f(x+iy)$

Each of real numbers u and v depends on real variable x and y and it follows that $f(z)$ can be expressed interms of a pair of real valued functions of real variable x and y

$f(z) = u(x,y) + iv(x,y)$

Let (r, θ) be the polar co-ordinates,

then $u+iv = f(re^{i\theta})$

where $w = u+iv$ and $z = re^{i\theta}$

So,

$f(z) = u(r,\theta) + iv(r,\theta) \rightarrow$ ②

3. Express the function $f(z) = z^2$ in form of polar co-ordinates :

Sol:-

Given that $f(z) = z^2$

Let $z = x+iy$

we know that $w = f(z)$

$\Rightarrow f(x+iy) = (x+iy)^2$

$= x^2 + i^2y^2 + 2xiy$

$= x^2 - y^2 + 2xiy$

Here $u = x^2 - y^2$

$v = 2xy$

In polar co-ordinates WKT $\Rightarrow z = re^{i\theta}$

$\Rightarrow f(re^{i\theta}) = (re^{i\theta})^2$

$= r^2 (e^{i\theta})^2$

$= 2\ i2\theta$

$= r^2 [\cos 2\theta + i \sin 2\theta]$

$= r^2 \cos 2\theta + i r^2 \sin 2\theta$

$\Rightarrow u(r, \theta) = r^2 \cos 2\theta$ and $v = r^2 \sin 2\theta$.

NOTE :-

If in both the equations ① and ② the function v always has a value zero, then the value of f is real. So, if f is a real valued function of a complex variable for

eg,

$$f(z) = |z|^2 = (\sqrt{x^2+y^2})^2 = [(x^2+y^2)^{1/2}]^2$$

$$= x^2+y^2+i0$$

$$\therefore f = x^2+y^2$$

So, f is a real valued function.

polynomial and Rational functions:

If "n" is a zero (or) a positive integer and $a_0, a_1, a_2, \ldots a_n$ are complex constants where $a_n \neq 0$, the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$ is a

polynomial of degree n. Note that the sum here a finite number of terms and that the domain of definition is the entire z plane.

Quotients $p(z) / Q(z)$ of polynomials are called rational and are defined at each point z where $Q(z) \neq 0$ polynomials and rational functions are elementary but important classes of function of a complex variable.

Single valued function:-

A single valued function is a function that if for one value of w, there corresponds a each value of z.

eg:-

$$f(z) = z^2$$

Multiple valued function:-

It is a rule that assigns more than one value to a point z in the domain of definition.

eg:-

$$f(z) = z^{1/2}$$

Problems:-

1) For each of functions below, describe the domain of definition.

a) $f(z) = \frac{1}{z^2+1}$

Sol:-

Given that

$$f(z) = \frac{1}{z^2+1}$$

f(z) is not defined when $z = \pm i$ $\forall z \in \mathbb{C}$

(or) $z^2+1 = 0$

$z^2 = -1$

$z = \pm i$

$\therefore$ Domain of definition is given by

$z = \pm i$

b) $f(z) = \arg(1/z)$

wkt $\Rightarrow z = x+iy$

Now $\frac{1}{z} = \frac{1}{x+iy}$

$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{x-iy}{(x+iy)(x-iy)}$

$= \frac{x-iy}{x^2-i^2y^2}$

$= \frac{x-iy}{x^2+y^2}$

$\Rightarrow f(z) = \arg\left(\frac{1}{z}\right)$

$= \arg\left(\frac{x-iy}{x^2+y^2}\right)$

$= \arg\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$

$= \tan^{-1}\left(\frac{-y/x^2+y^2}{x/x^2+y^2}\right)$

$f(z) = \tan^{-1}(-y/x)$

$\therefore$ Domain of define is given by $x \neq 0$

c) $f(z) = \frac{z}{z+\bar{z}}$

Sol:-

WKT $\Rightarrow z = x+iy$

and $\bar{z} = x-iy$

$\Rightarrow f(z) = \frac{x+iy}{x+iy+x-iy}$

$= \frac{x+iy}{2x}$

$\therefore$ Domain of definition is $Re\ z \neq 0$.

d) $f(z) = \frac{1}{1-|z|^2}$

Sol:-

The function f is not defined when

$z = \pm 1$

Domain of definition is given by

$z \neq \pm 1$

2) write the function $f(z) = z^3+z+1$ in the form $f(z) = u(x,y)+iv(x,y)$

sol:-

WKT $\Rightarrow z = x+iy$

$f(z) = z^3+z+1$

$= (x+iy)^3 + (x+iy)+1$

$= x^3 - iy^3 + 3x^2y\,i - 3y^2x + x + iy + 1$

$= x^3 + 3x^2y\,i - 3y^2x + x + iy + 1 - iy^3$

$f(z) = (x^3 - 3y^2x + x + 1) + i(3x^2y + y - y^3)$

3. Write the function $f(z) = z + 1/z \ (z \neq 0)$ in the form $f(z) = u(r,\theta) + iv(r,\theta)$.

Sol:-

$$f(z) = z + 1/z$$

$$z = re^{i\theta}$$

$$= re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

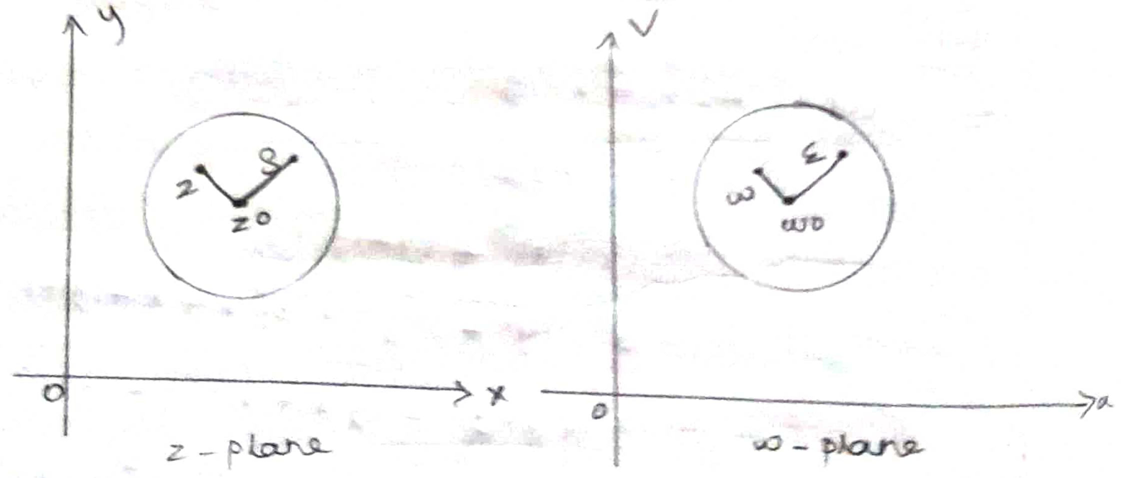
$$= \left(r+\frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\sin\theta\right)$$

Limit of A function:-

Let S be a set of complex numbers. Let f be a function defined at all points z in some deleted neighborhood of z_0. Then said to have a limit w_0 if for each positive number ε, there is a positive number δ, such that

$$|f(z) - w_0| < \varepsilon \ \forall z \in S \text{ whenever } 0 < |z - z_0| < \delta$$

ie $\lim_{z \to z_0} f(z) = w_0$



Theorem : 1

prove that the limit of a function is unique.

proof :

Let us assume that $\lim_{z \to z_0} f(z) = w_0$ and $\to ①$

$$\lim_{z \to z_0} f(z) = w_1 \to ②$$

Then by definition of limits,

① $\to$ for each positive number ε_1 there is a positive number δ_0 such that

$$|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta_0$$

and

② $\to$ for each positive number ε_1 there exist a $\delta_1 > 0$ such that $|f(z) - w_1| < \varepsilon$ whenever $0 < |z - z_0| < \delta_1$

Consider,

$$|w_1 - w| = |w - f(z) + f(z) - w_0|$$

$$= |-(f(z) - w_1) + |f(z) - w_0|$$

$$\leq |-f(z) - w_1)| + |f(z) - w_0|$$

$$< \varepsilon + \varepsilon$$

$$< 2\varepsilon$$

Since ε is arbitary small we have

$\Rightarrow |w_1 - w_0| = \varepsilon$ ($\because |w_1 - w_0|$ is non-negative constant)

$\Rightarrow w_1 - w_0 = 0$

$\Rightarrow w_1 = w_0$

$\therefore$ The limit of a function is unique.

1. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ where $z = x+iy$ use the expression $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ to write $f(z)$ interms of z and simplify the result.

Sol:-

$$f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$$

$$= \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 - 2\left(\frac{z-\bar{z}}{2i}\right) + i$$

$$F(z) = x^2 - y^2 - 2y + i(2x - 2xy)$$

$$= \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 - 2\left(\frac{z-\bar{z}}{2i}\right) + i\left[2\left(\frac{z+\bar{z}}{2}\right)\right.$$
$$\left. 2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)\right]$$

$$= \frac{z^2 + \bar{z}^2 + 2z\bar{z} + z^2 + \bar{z}^2 - 2z\bar{z}}{4} - \left(\frac{z-\bar{z}}{i}\right) +$$
$$i\left[z+\bar{z} - 2\left(\frac{z^2-\bar{z}^2}{4i}\right)\right]$$

$$= \frac{2z^2 + z\bar{z}^2}{4} - \left(\frac{z-\bar{z}}{i}\right) + i\left(\frac{2iz + 2i\bar{z} - z^2 + \bar{z}^2}{2i}\right)$$

$$= \frac{z^2 + \bar{z}^2}{2} - \left(\frac{z-\bar{z}}{i}\right) +$$

$$= \frac{z^2 + \bar{z}^2 + 2iz + 2i\bar{z} - z^2 + \bar{z}^2}{2} - \left(\frac{z-\bar{z} \times i}{i \times i}\right)$$

$$= \frac{2\bar{z}^2 + 2iz + 2i\bar{z} - \cancel{z^2} + \cancel{z^2}}{2} + zi - \bar{z}$$

$$= \frac{2\bar{z}^2 + 2iz + 2i\bar{z}}{2} + zi - \bar{z}i$$

$$= 2\left(\frac{\bar{z} + iz + i\bar{z}}{2}\right) + zi - \bar{z}i$$

$$= \bar{z}^2 + iz + i\bar{z} + zi - \bar{z}i$$

$$\therefore f(z) = \bar{z}^2 + 2iz$$

Problems:-

1. Show that if $f(z) = \frac{i\bar{z}}{2}$ in open disc $|z|<1$, then $\lim_{z \to 1} f(z) = i/2$, the point, being on boundary of domain of definition of f.

Sol:-

Gn $\Rightarrow f(z) = \frac{i\bar{z}}{2}, \quad |z|<1 \to (1)$

we have to prove that $\lim_{z \to 1} = i/2$

ie) To prove that if z approaches to 1

$f(z)$ tends to $i/2$

ie) for each positive number ε there exist a $\delta > 0$, such that

$|f(z) - i/2| < \varepsilon \quad \forall z \in S$, whenever $0<|z-1|<\delta$

consider

$|f(z) - i/2| \quad |\ i$

$\leq \left|\frac{i}{2}\right| |z-1|$

$\leq \frac{|i|}{2} |z-1|$

$< \frac{1}{2} |z-1|$

Now, let us choose $\delta = 2\varepsilon$

$< \frac{1}{2}(2\varepsilon) < \varepsilon$

$\Rightarrow |f(z) - i/2| < \varepsilon$ whenever $0<|z-1|<\delta$

$\lim_{z \to 1}, f(z) = i/2$

$\therefore$ Hence proved.

2. If $f(z) = \frac{z}{\bar{z}}$, then $\lim_{z \to 0} f(z)$ does not exist.

proof:

Given that

$$f(z) = \frac{z}{\bar{z}}$$

To prove that $\lim_{z \to 0} f(z)$ does not exist

we know that $z = x+iy$, $\bar{z} = x-iy$

$$f(z) = \frac{x+iy}{x-iy}$$

Let $z = (x,y)$ be a point. It approaches to origin in any manner. But when

$z = (x,0)$ is a non-zero point on real axis

$$f(z) = \frac{x+0i}{x+0i} = \frac{x}{x} = 1$$

when $z = (0,y)$ in a non-zero point on Imaginary axis

$$\Rightarrow f(z) = \frac{yi}{-yi} = -1$$

Here, the limit is not unique

So, $\lim_{z \to 0} f(z)$ does not exist

THEOREM ON LIMITS :-

theorem:-

Suppose that $f(z) = u(x,y) + iv(x,y)$.

$z = x+iy$ and $z_0 = x_0+iy_0$, $w_0 = u_0+iv_0$, then

$\lim_{z \to z_0} f(z) = w_0$ if and only if

$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x,y)$

proof :-

Let f be a function defined on a set S and let z_0 be a limit point on S

1st let us assume that

$$\lim_{z \to z_0} f(z) = w_0 \rightarrow (1)$$

To prove that

$$\lim_{(x,y) \to (x_0, y_0)} u(x,y) = u_0 \text{ and}$$

$$\lim_{(x,y) \to (x_0, y_0)} v(x,y) = v_0$$

By defn,

for each positive no ε, there is a positive number δ, such that

$$|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

ie $|u(x,y) + iv(x,y) - (u_0 + iv_0)| \varepsilon$ whenever

$$0 < |x+iy - (x_0 + iy_0)| < \delta.$$

Consider

$$|z - z_0| = |(x+iy) - (x_0 + iy_0)|$$

$$= |x - x_0 + iy - iy_0|$$

$$= |x - x_0 + i(y - y_0)|$$

$$= \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Now consider,

$$|(u+iv) - (u_0 + iv_0)| < \varepsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

$$\Rightarrow |u - u_0 + iv - iv_0| < \varepsilon \text{ whenever}$$

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

$\Rightarrow |(u-u_0)+i(v-v_0)| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

$\Rightarrow \sqrt{(u-u_0)^2+(v-v_0)^2} < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

$\Rightarrow (u-u_0)^2+(v-v_0)^2 < \varepsilon^2$ whenever $0 < (x-x_0)^2 + (y-y_0)^2 < \delta^2$

$\Rightarrow (u-u_0)^2 < \varepsilon^2 ; (v-v_0)^2 < \varepsilon^2$ whenever $0 < (x-x_0)^2 < \delta^2$

$0 < (y-y_0)^2 < \delta^2$

$\Rightarrow |u-u_0| < \varepsilon ; |v-v_0| < \varepsilon ;$ whenever $0 < |x-x_0| < \delta$

$0 < |y-y_0| < \delta$

$\Rightarrow |u-u_0| < \varepsilon ; |v-v_0| < \varepsilon$ whenever $0 < |z-z_0| < \delta$.

$\Rightarrow \lim_{(x,y) \to (x_0, y_0)} u(x,y) = u_0$ and

$\lim_{(x,y) \to (x_0, y_0)} v(x,y) = v_0$

conversely assume that

$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \rightarrow ①$ $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0 \rightarrow ②$

To prove that $\lim_{z \to z_0} f(z) = w_0$

By defn ① & ② $\Rightarrow$

for each positive number ε, there is a positive number δ_1 & δ_2 such that,

$\Rightarrow |u-u_0| < \varepsilon/2$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1$

$\Rightarrow |v-v_0| < \varepsilon/2$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$

consider

$|f(z) - w_0| = |u+iv - (u_0+iv_0)|$

$\leq |u - u_0 + i(v - v_0)|$

$< \varepsilon/2 + \varepsilon/2$

$< \varepsilon$

and $\sqrt{(x-x_0)^2 + (y-y_0)^2} = |(x-x_0) + i(y-y_0)|$

$= |(x+iy) - (x_0 + iy_0)|$

$= |z - z_0|$

$\therefore |f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

$\Rightarrow \lim_{z \to z_0} f(z) = w_0$ whenever $0 < |z - z_0| < \delta$

$\therefore$ Hence proved.

Theorem : 2

suppose that $\lim_{z \to z_0} f(z) = w_0$ and

$\lim_{z \to z_0} F(z) = W_0$

Then

i) $\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0$

ii) $\lim_{z \to z_0} [f(z) F(z)] = w_0 W_0$

iii) $\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$ if $W_0 \neq 0$

proof :-

i) given $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$

By defn

for each positive number ε, there is a positive numbers, δ_1 & δ_2 such that

$|f(z) - w_0| < \varepsilon/2$ whenever $0 < |z - z_0| < \delta_1$

and

$|F(z) - W_0| < \varepsilon/2$ whenever $0 < |z - z_0| < \delta_2$

consider,

$$|f(z) + F(z) - (w_0 + W_0)| = |f(z) + F(z) - w_0 - W_0|$$

$$= |f(z) - w_0 + F(z) - W_0|$$

$$\leq |f(z) - w_0| + |F(z) - W_0|$$

$$< \varepsilon/2 + \varepsilon/2 ; \text{ choose } \delta = \min\{\delta_1, \delta_2\}$$

$$< \varepsilon ; \quad 0 < |z - z_0| < \delta$$

$$\lim_{z \to z_0} (f(z) + F(z))$$

$$= w_0 + W_0$$

Hence proved

ii) given

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} F(z) = W_0$$

By definition,

for each positive number ε, there is a positive numbers, δ_1 & δ_2 such that

$$|f(z) - w_0| < \varepsilon/2k \text{ whenever } 0 < |z - z_0| < \delta_1$$

and

$$|F(z) - W_0| < \varepsilon/2w_0 \text{ whenever } 0 < |z - z_0| < \delta_2$$

Consider,

$$|f(z)F(z) - w_0 W_0| = |f(z)F(z) - w_0 F(z) + w_0 F(z) - w_0 W_0|$$

$$= |F(z)[f(z) - w_0] + w_0(F(z) - W_0)|$$

$$\leq |F(z)||f(z) - w_0| + |w_0||F(z) - W_0|$$

Now

Given that $\lim_{z \to z_0} F(z) = W_0$

ie) $|F(z) - W_0| < 1 \; ; \; 0 < |z - z_0| < \delta_3$

let $|F(z)| = |F(z) - W_0 + W_0|$

$\leq |F(z) - W_0| + |W_0|$

$< 1 + |W_0|, \; 0 < |z - z_0| < \delta_3$

let $K = 1 + |W_0|$

$\Rightarrow |F(z)| < K, \; 0 < |z - z_0| < \delta_3$

chaose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

Now,

$$|f(z)F(z) - w_0 W_0| < K \cdot \frac{\varepsilon}{2K} + |w_0| \cdot \frac{\varepsilon}{2|w_0|}$$

$$< \varepsilon/2 + \varepsilon/2$$

$$< \varepsilon \; ; \; 0 < |z - z_0| < \delta$$

$$\lim_{z \to z_0} (f(z)F(z)) = w_0 W_0$$

iii) Given that

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} F(z) = W_0$$

for each positive number ε, there is a positive number δ_1 & δ_2 such that

$$|f(z) - w_0| < \frac{|W_0|^2 \cdot \varepsilon}{4} \text{ whenever } 0 < |z - z_0| < \delta_1$$

and $|F(z) - W_0| < \dfrac{|W_0|^2 \cdot \varepsilon}{|w_0| \cdot 4}$ whenever $0 < |z - z_0| < \delta_2$

Now consider,

$$\left| \frac{f(z)}{F(z)} - \frac{w_0}{W_0} \right| = \left| \frac{f(z) W_0 - w_0 F(z) + W_0 W_0 - w_0 W_0}{F(z) \cdot W_0} \right|$$

$$= \frac{1}{|F(z)||W_0|} \, |W_0 [f(z) - w_0] - w_0 (F(z) - W_0)|$$

$$\leq \frac{1}{|F(z)||W_0|} \, |W_0||f(z) - w_0| + |w_0||F(z) - W_0|$$

since, $\lim_{z \to z_0} F(z) = w_0 (\neq 0)$

there exist a δ_3 such that,

$|F(z) - w_0| < \frac{1}{2} |w_0|$ whenever $0 < |z - z_0| < \delta_3$

$\Rightarrow |F(z)| > \frac{1}{2} |w_0|$ for $0 < |z - z_0| < \delta_3$

ie) $\frac{1}{2} |w_0| < |F(z)|$

Let $\delta = \min \{ \delta_1, \delta_2, \delta_3 \}$

$$\Rightarrow \left| \frac{f(z)}{F(z)} - \frac{w_0}{w_0} \right| < \frac{1}{\frac{|w_0|}{2} \cdot |w_0|} \left[\frac{|w_0| \cdot \varepsilon |w_0|}{4} + \frac{|w_0| \cdot |w_0|^2 \cdot \varepsilon}{|w_0| \cdot 4} \right]$$

$$< \frac{2}{|w_0|^2} \left[\frac{\varepsilon \cdot |w_0|^2}{4} + \frac{\varepsilon |w_0|^2}{4} \right]$$

$$< \frac{2}{|w_0|^2} \cdot \frac{|w_0|^2}{2} \left[\varepsilon/2 + \varepsilon/2 \right]$$

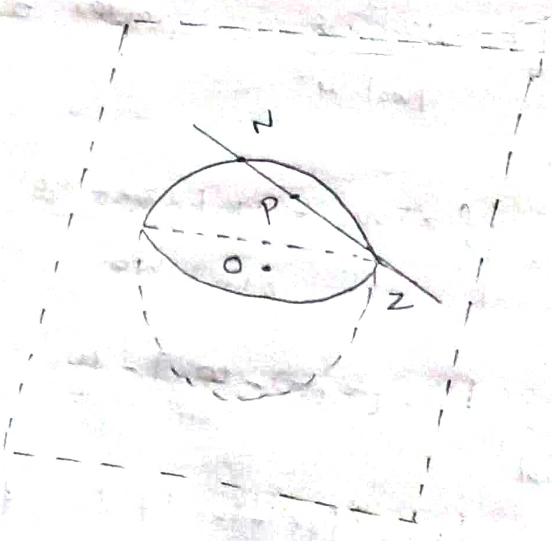
$$< \varepsilon/2 + \varepsilon/2$$

$$\left| \frac{f(z)}{F(z)} - \frac{w_0}{w_0} \right| < \varepsilon$$

$$\lim_{z \to z_0} \left(\frac{f(z)}{F(z)} \right) = \frac{w_0}{w_0}$$

$\therefore$ Hence proved.

Limits Involving the point at infinity :-



It is sometimes convinent of include with the complex plane at infinity, denoted by ∞ and to use the limits involving it.

The complex plane together with this point (e) the point at infinity is called the extended complex plane.

To visualize the point at infinity, think the complex x plane as passing through the equator of the units sphere centered at the origin.

To each point z in the complex plane, the corresponds exactly one point p on the surface. The point p is the point where the line through z and the north pole N intersects the sphere in like manner, to each point p on the surface of sphere, other than the north pole N, there corresponds exactly one point z in the plane.

By letting the point N of the sphere correspond to the point at infinity there is a one-to-one correspondence between the points on the sphere and the points of the extended complex plane.

This sphere is known as riemann sphere and the correspondence is called a stereographic projection.

Observe that the exterior of the unit circle centered at the origin in the complex plane corresponds to the upper hemisphere with the equator and the point N deleted.

Moreover for each small positive number ε, those points in the complex plane exterior to the circle $|z| = 1/\varepsilon$ corresponds to points on sphere close to N. The set $|z| > 1/\varepsilon$ is called an ε-neighborhood (or) neighborhood of ∞.

Theorem :- (Limit at infinity)

If z_0 and w_0 are the points in z and w planes respectively, then prove that

i) $\lim_{z \to z_0} f(z) = \infty$ iff $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

ii) $\lim_{z \to \infty} f(z) = w_0$ iff $\lim_{z \to 0} f(1/z) = w_0$

iii) $\lim_{z \to \infty} f(z) = \infty$ iff $\lim_{z \to 0} \frac{1}{f(1/z)} = 0$

Proof :-

i) Let us assume that

$$\lim_{z \to z_0} f(z) = \infty \rightarrow (1)$$

To prove that

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0$$

(1) $\Rightarrow$ for each positive number ε, there is a positive number δ, such that,

$$|f(z)| > 1/\varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

ie) the point $w = f(z)$ lies in ε-neighborhood $|w| > 1/\varepsilon$ of ∞ whenever z lies in the neighborhood of z_0

$$\therefore \left|\frac{1}{f(z)}\right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

$$\left|\frac{1}{f(z)} - 0\right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

$$\therefore \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

$\therefore$ Hence proved

conversely assume that $\lim_{z \to z_0} \frac{1}{f(z)} = 0.$

To prove that : $\lim_{z \to z_0} f(z) = \infty$

Since,

$$\lim_{z \to z_0} \left(\frac{1}{f(z)}\right) = 0$$

By defn,

for each positive number ε, there is a positive number δ, such that

$$\left|\frac{1}{f(z)} - 0\right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

$$\Rightarrow \left|\frac{1}{f(z)} < \varepsilon\right|$$

$$\Rightarrow |f(z)| > 1/\varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

$$\therefore \lim_{z \to z_0} f(z) = \infty$$

ii) 1st let us assume that

$$\lim_{z \to \infty} f(z) = w_0 \rightarrow ①$$

To prove that : $\lim_{z \to 0} f(1/z) = w_0$

① $\rightarrow$ for each positive number ε, there is a positive number δ, such that

$$|f(z) - w_0| < \varepsilon \text{ whenever } |z| > 1/\delta$$

Replacing z by $1/z$,

$$|f(1/z) - w_0| < \varepsilon \text{ whenever } 0 < |z - 0| < \delta$$

Hence,

$$\lim_{z \to 0} f(1/z) = w_0$$

Conversely assume that $\lim_{z \to 0} f(1/z) = w_0$

To prove that : $\lim_{z \to \infty} f(z) = w_0$

for each positive number ε, there is a positive number δ, $\ni$

$$|f(1/z) - w_0| < \varepsilon \text{ whenever } 0 < |z-0| < \delta$$

$$\Rightarrow |f(1/z) - w_0| < \varepsilon \text{ whenever } 0 < |z| < \delta$$

Replacing $1/z$ by z

$$|f(z) - w_0| < \varepsilon \text{ whenever } |1/z| < \delta \Rightarrow |z| > 1/\delta$$

$$\lim_{z \to \infty} f(z) = w_0$$

Assume that

$$\lim_{z \to \infty} f(z) = \infty$$

iii) To prove that

$$\lim_{z \to 0} \frac{1}{f(1/z)} = 0$$

$\Rightarrow$ for each positive number ε, there is a positive number δ, such that

$$|f(z)| > \frac{1}{\varepsilon} \text{ whenever } |z| > 1/\delta$$

replacing z by $1/z$

$$\Rightarrow \left| \frac{1}{f(1/z)} - 0 \right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

$$\therefore \lim_{z \to \infty} \frac{1}{f(1/z)} = 0$$

conversely assume that

$$\lim_{z \to 0} \frac{1}{f(1/z)} = 0$$

To prove that

$$\lim_{z \to \infty} f(z) = \infty$$

→ for $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$\left| \frac{1}{f(1/z)} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta$$

$$\left| \frac{1}{f(z)} \right| < \varepsilon \Rightarrow \frac{1}{|z|} < \delta$$

$$|f(z)| > \varepsilon, \quad |z| > 1/\delta$$

$$\lim_{z \to 0} f(z) = \infty$$

Examples:-

1. $\lim_{z \to -1} \frac{iz+3}{z+1} = \infty$ since $\lim_{z \to -1} \frac{z+1}{iz+3} = 0$

2. $\lim_{z \to \infty} \frac{2z+i}{z+1} = 2$ since $\lim_{z \to 0} \frac{(2/z)+i}{(1/z)+1} = \lim_{z \to 0} \frac{2+iz}{1+z} = 2$

3. $\lim_{z \to \infty} \frac{2z^3-1}{z^2+1} = \infty$ since $\lim_{z \to 0} \frac{(1/z^2)+1}{(2/z^3)+1} = \lim_{z \to 0} \frac{z+z^3}{2-z^3} = 0$

continuity:-

A function f is continuous at a point z_0 if all three of following conditions are satisfied.

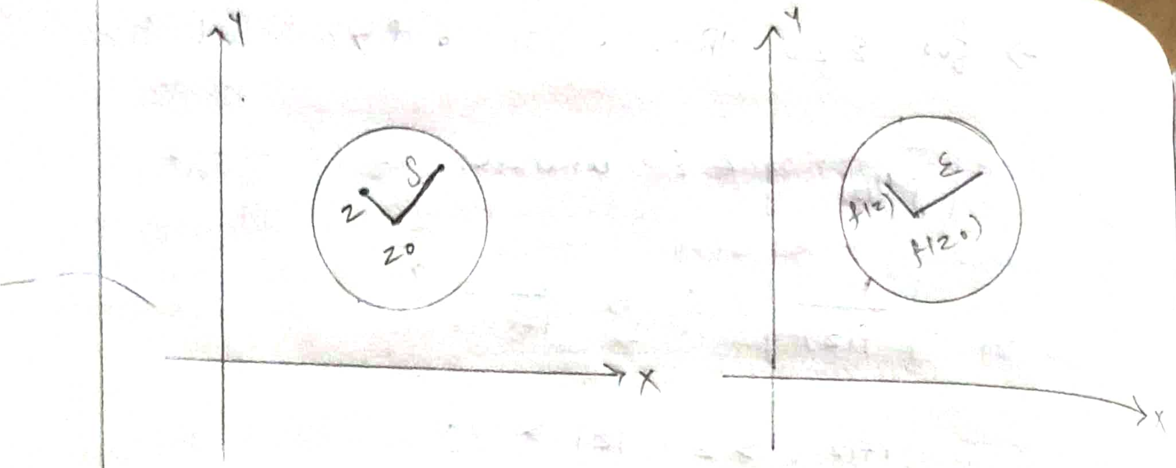
i) $\lim_{z \to z_0} f(z)$ exist

ii) $f(z_0)$ exist

iii) $\lim_{z \to z_0} f(z) = f(z_0)$

ie) for each positive number ε, there is a positive number δ, such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta$$



Continuity in a region R:-

A function of a complex variable is said to be continuous in a region R if it is continuous at each point in R.

Note:-

If two functions are continuous at a point, then sum and product are also continuous.

If the denominators is not zero, their Quotient is continuous.

Examples:-

1. Check whether the function $f(z) = \begin{cases} z^2 & \text{if } z \neq i \\ 0 & z = i \end{cases}$ is continuous or not.

Sol:-

Given that,

$$f(z) = \begin{cases} z^2 & \text{if } z \neq i \\ 0 & \text{if } z = i \end{cases}$$

By defn,

$$\lim_{z \to i} f(z) = i^2$$
$$= -1.$$

But if $z = i$, Then function is zero

ie) $\lim_{z \to i} f(x) = 0$

$\therefore$ The given function is not continuous.

2. $\lim_{z \to 2} \frac{z^2-4}{z-2} = 4$ is not continuous at $z=2$

3. $f(z) = z^2$ is continuous.

Theorem:

prove that the composition of a continuous function is itself continuous.

proof:-

Let S be a set of complex numbers and z_0 be a limit point of S contained in S.

Let $w = f(z)$ be a function defined for all z in a neighborhood $|z-z_0| < \delta$ of a point z_0.

Let $W = g(w)$ be a function whose domain of definition contains the image of that neighborhood under f.

Suppose that if $g(z)$ is defined on S, $g(z)$ is continuous at (z_0).

To prove that $g \circ f$ is continuous at z_0

Since g is continuous at $f(z_0)$ in w-plane.

By defn,

for each positive number ε, there is a positive number δ, such that

$|g[f(z)] - g[f(z_0)]| < \varepsilon$ whenever $|f(z)-f(z_0)| < \delta$

Since $f(z)$ is continuous at z_0, then by definition for each positive number ε, there is a positive number δ, such that

$|f(z) - f(z_0)| < \varepsilon$, whenever $0 < |z - z_0| < \delta$

choose $\varepsilon_1 = \eta$

Then,

(2) $\to |f(z) - f(z_0)| < \delta$ whenever $|z - z_0| < \delta \to$ (2)

from (1) & (2)

$|g[f(z)] - g[f(z_0)]| < \varepsilon$ whenever $|z - z_0| < \delta$

(or)

$|(g\circ f)z - (g\circ f)(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$

∴ The composition of continuous function is continuous.

∴ Hence proved

Theorem :-

If a function $f(z)$ is continuous and non-zero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.

Proof :-

Let us assume that $f(z)$ is continuous and non-zero at a point z_0

To prove that $f(z) \neq 0$ throughout some neighborhood of that point.

Since f is continuous at z_0, by defn for each positive number ε, there is a positive number δ, ∋

$|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta \to$ (1)

Let

$$\varepsilon = \frac{|f(z_0)|}{2}$$

Then,

① $\rightarrow |f(z)-f(z_0)| < \frac{|f(z_0)|}{2}$ with $|z-z_0| < \delta$

So, if there is a point z in neighborhood $|z-z_0|<\delta$ at which $f(z)=0$, we have the contradiction

$$|f(z_0)| < \frac{|f(z_0)|}{2}$$

$\therefore$ $f(z)$ is not zero throughout some neighborhood at that point.

NOTE :-

i) If continity of a function the $f = u(x,y)+iv(x,y)$ is closed related to continuity of its component function.

ie) The function f is continuous at a point (x_0, y_0) if and only if its components function are continuous there.

ii) A region R is closed if it contains all of its boundary point and that is bounded if it lies inside some circle centered that origin

Theorem :-

If a function f is continuous throughout the region R that is both bounded, and closed there exist a non-negative (M) real number such that $|f(z)| \leq M$ for all points z in R, where the equality holes for atleast one such z.

proof :-

Given that function f is continuous throughout the region R

W.K.T $\Rightarrow f(z) = u(x,y) + iv(x,y)$

$$|f(z)| = |u(x,y) + iv(x,y)|$$

$$= \sqrt{u(x,y)^2 + (v(x,y))^2}$$

The function $\sqrt{(u(x,y))^2 + (v(x,y))^2}$ is continuous throughout the region R. Let M be a maximum value of square root

$\therefore$ The above function, thus reaches the maximum value M

$$|f(z)| \leq M$$

$\therefore f$ is bounded in R.

Derivatives:-

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0, then the derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \to ①$$

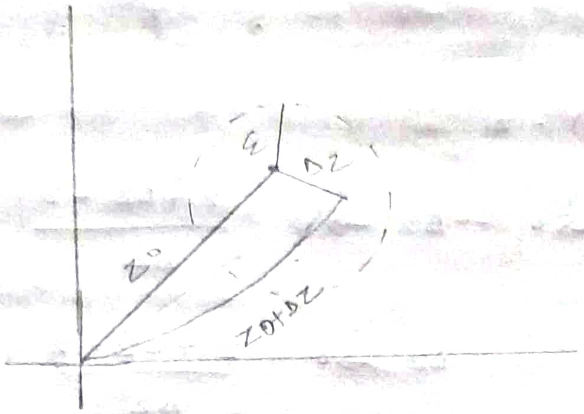
and the function f is said to be differentiable at z_0 when $f'(z_0)$ exist.

If, $\Delta z = z - z_0 \quad (z \neq z_0)$

Then,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta} \to ②$$

f is defined throughout a neighborhood of z_0, the number $f(z_0 + \Delta z)$ is always defined for $|\Delta z|$ sufficiently small.



we drop the subscript on z_0 we get

$$\Delta w = f(z+\Delta z) - f(z)$$

which denotes the change in the value $w=f(z)$ of f corresponding to change Δz in the point at which f is evaluated

io) $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$

ie) $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$

Eg:-

1. prove that $f(z) = z^2$ is differentiable at any point z.

proof :-

gn $\Rightarrow f(z) = z^2$

W.K.T $\Rightarrow \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{z^2 + (\Delta z)^2 + 2z.\Delta z - z^2)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\Delta z)^2 + 2z.\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(\Delta z + 2z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (\Delta z + 2z)$$

$$= 2z + 0 \quad = 2z$$

$\therefore f'(z) = 2z$

2. $f(z) = \bar{z}$

Sol:-

Let $f(z) = \bar{z}$

w.k.T, $\lim_{\Delta z \to 0} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\bar{z} + \Delta\bar{z} - \bar{z}}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\Delta\bar{z}}{\Delta z}$

By letting

$\Delta z = z - z_0$

$= (x+iy) - (x_0 + iy_0)$

$= x + iy - x_0 - iy_0$

$= (x - x_0) + i(y - y_0)$

Now,

Δz must approach to the point of origin in any manner.

If it approaches along real axis ie $(\Delta x, 0)$

$\Delta z = (x - x_0) + i0$

$\Rightarrow \overline{\Delta z} = \overline{\Delta x + i0}$

$= \Delta x - i0$

$= \Delta x + i0$

$\Delta\bar{z} = \Delta z$

$\therefore \frac{\Delta w}{\Delta z} = \frac{\Delta z}{\Delta z} = 1$

Hence limits $\frac{\Delta w}{\Delta z}$ exist & is value will be unity.

Now, if it approaches along Imaginary axis, then

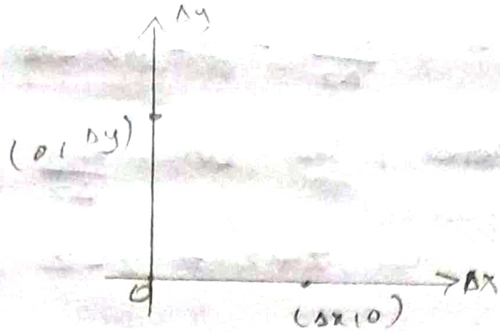
$\Delta z = 0 + i(y - y_0)$

$\Delta\bar{z} = \overline{0 + i\Delta y}$

$= 0 - i\Delta y$

$= -\Delta z$

$\frac{\Delta w}{\Delta z} = \frac{-\Delta z}{\Delta z} = -1$



Since, the limits are unique $\frac{dw}{dz}$ does not exist.

3. consider the real valued function $f(z) = |z|^2$ find the limit if it exist or not.

Sol:-

given $\Rightarrow f(z) = |z|^2$

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\Delta \bar{z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\bar{z} + z\Delta\bar{z} + \Delta z.\bar{z} + \Delta z\,\Delta\bar{z} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\Delta\bar{z} + \Delta z\,\bar{z} + \Delta z.\Delta\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} z\frac{\Delta\bar{z}}{\Delta z} + \bar{z} + \Delta\bar{z}$$

$$= \lim_{\Delta z \to 0} \bar{z} + \Delta\bar{z} + z\frac{\Delta\bar{z}}{\Delta z}$$

we know that, $\Delta\bar{z} = \Delta z$ and $\Delta\bar{z} = -\Delta z$ [By previous Example]

$\frac{\Delta w}{\Delta z} = \bar{z} + \Delta z + z$ when $\Delta z = (\Delta x, 0)$

and $\frac{\Delta w}{\Delta z} = \bar{z} - \Delta z - z$ when $\Delta z = (0, \Delta y)$

Hence, the $\lim \frac{\Delta w}{\Delta z}$ exist as $\Delta z \to 0$

ie) $\bar{z} + z = \bar{z} - z$ (or) $z = 0$

when $z \neq 0$, $\frac{dw}{dz}$ does not exist, when $z = 0$

ie) $\frac{\Delta w}{\Delta z} = \lim_{z \to 0} \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z} = \overline{\Delta z}$

So, $\frac{dw}{dz}$ exist when $z = 0$ & value of is zero.

Note :

The function $f(z) = u(x, y) + iv(x, y)$ can be differentiable at $z = (x, y)$ but nowere else in any neighborhood of that point since $f(z) = |z|^2$

ie) $u(x, y) = x^2 + y^2$ $\{v(x, y)\} = 0$

It also shows that the real and Imaginary components of a function of a complex variable can have its partial derivative of all order at $z = (x, y)$ and yet the function is not differentiable.

Theorem:-

Necessary condition for differentiabilities :-

Statement :-

If a function $f(z)$ is differentiable at z_0, Then $f(z)$ is continuous at that point

(or)

The existence of derivative of a function at a point implies the continuity at that point.

proof :-

Consider the function $f(z)$ is differentiable at z_0.

By defn,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

To prove that $f(z)$ is continuous consider

$$\lim_{z \to z_0} [f(z)-f(z_0)] = \lim_{z \to z_0} \left[f(z)-f(z_0) \times \frac{z-z_0}{z-z_0} \right]$$

$$= \lim_{z \to z_0} \left[\frac{f(z)-f(z_0)}{z-z_0} \times (z-z_0) \right]$$

$$= \lim_{z \to z_0} \left[\frac{f(z)-f(z_0)}{z-z_0} \right] \times \lim_{z \to z_0} (z-z_0)$$

$$\lim_{z \to z_0} [f(z)-f(z_0)] = f'(z_0) \times 0$$

$$= 0$$

$$\lim_{z \to z_0} f(z) = f(z_0)$$

The function f is continuous.

$\therefore$ Hence proved.

NOTE :-

The converse of the above theorem is not true.

Differentiation formulas:-

The derivative of a function f at a point z_1 is denoted by

$$\frac{d}{dz} f(z) \text{ (or) } f'(z)$$

ie) Let c be a complex constants and let f be a function whose derivative exist at a point z_1

ie)

$$\frac{d}{dz}[c] = 0$$

$$\frac{d}{dz}[z] = 1$$

$$\frac{d}{dz}[cf(z)] = cf'(z)$$

If n is a positive integer, then

$$\frac{d}{dz}[z^n] = nz^{n-1}.$$

This formula remains valid when $z \neq 0$.

Theorem :-

If the derivatives of two function f and g exist at a point z, then.

1) $\frac{d}{dz}[f(z)+g(z)] = f'(z)+g'(z)$.

proof :-

Given that f and g are differentiable.

Then By defn,

we know that

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

and

$$g'(z) = \lim_{\Delta z \to 0} \frac{g(z+\Delta z)-g(z)}{\Delta z}$$

Now,

$$\frac{d}{dz}[f(z)+g(z)] = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)+g(z+\Delta z) - (f(z)+g(z))}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)+g(z+\Delta z)-f(z)-g(z)}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)-f(z)+g(z+\Delta z)-g(z)}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right] + \lim_{\Delta z \to 0} \left[\frac{g(z+\Delta z)-g(z)}{\Delta z}\right]$$

$$= f'(z)+g'(z)$$

$$\therefore \frac{d}{dz}[f(z)+g(z)] = f'(z)+g'(z)$$

$\therefore$ Hence proved.

ii) $\frac{d}{dz}[f(z).g(z)] = f(z)g'(z) + g(z)f'(z)$

proof :-

Given that f and g are differentiable

Then By defn,

we know that,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

and

$$g(z) = \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z}$$

Now,

$$\frac{d}{dz}[f(z).g(z)] = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)\, g(z+\Delta z) - f(z)g(z)}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)\, g(z+\Delta z) - f(z+\Delta z)g(z) + f(z+\Delta z)g(z) - f(z)g(z)}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)g(z+\Delta z) - f(z+\Delta z)g(z)}{\Delta z}\right] + \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)g(z) - f(z)\,g(z)}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z)\,[g(z+\Delta z) - g(z)]}{\Delta z}\right] + \lim_{\Delta z \to 0} \left[\frac{g(z)\,[f(z+\Delta z - f(z)]}{\Delta z}\right]$$

$$= f(z) \lim_{\Delta z \to 0} \left[\frac{g(z+\Delta z) - g(z)}{\Delta z}\right] + g(z) \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right]$$

$$= f(z)\,g'(z) + g(z) + f'(z)$$

∴ Hence proved,

(iii) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)\, f'(z) - f(z) g'(z)}{[g(z)]^2}$

Proof:-

Given that f and g are differentiable

Then By defn,

we know that,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

and

$$g'(z) = \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z}$$

Now,

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z)}{g(z+\Delta z)} - \frac{f(z)}{g(z)}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{\frac{f(z+\Delta z) g(z) - f(z) g(z+\Delta z)}{g(z+\Delta z)\, g(z)}}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z)\, g(z) - f(z) g(z+\Delta z)}{\Delta z\,[g(z+\Delta z) g(z)]}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) g(z) - f(z) g(z) + f(z) g(z) - f(z) g(z+\Delta z)}{\Delta z \cdot g(z+\Delta z)\, g(z)}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{g(z)\,[f(z+\Delta z) - f(z)] - f(z)\,[g(z+\Delta z) - g(z)]}{\Delta z \cdot g(z+\Delta z) g(z)}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{g(z)\,[f(z+\Delta z) - f(z)]}{\Delta z \cdot g(z+\Delta z)\, g(z)}\right] - \lim_{\Delta z \to 0}\left[\frac{f(z)\,[g(z+\Delta z) - g(z)]}{\Delta z \cdot g(z+\Delta z)\, g(z)}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right] - \lim_{\Delta z \to 0}\left[\frac{g(z)}{g(z+\Delta z)g(z)}\right] -$$

$$\lim_{\Delta z \to 0}\left[\frac{f(z)}{g(z+\Delta z)g(z)}\right] \lim_{\Delta z \to 0}\left[\frac{g(z+\Delta z)-g(z)}{\Delta z}\right]$$

$$= \frac{f'(z).g(z)}{[g(z)]^2} - \frac{f(z).g'(z)}{[g(z)]^2}$$

$$= \frac{g(z)f'(z)-f(z)g(z)}{[g(z)]^2}$$

∴ Hence proved.

NOTE :-

There is also a chain rule for differentiating composite functions. Suppose that f has a derivative at z_0 and g has a derivative at point $f(z_0)$. Then the function $F(z) = g[f(z)]$ has a derivative at z_0 and

$$F'(z_0) = g'[f(z_0)]\, f'(z_0)$$

If we write $w = f(z)$ and $W = g(w)$ so that $W = F(z)$, the chain rule becomes

$$\frac{dW}{dz} = \frac{dW}{dw} \cdot \frac{dw}{dz}$$

Example :-

Find the derivative of $(2z^2+i)^5$

Sol:-

Let $w = 2z^2+i$

Then $W = w^5$

$$\frac{dW}{dz} = \frac{dW}{dw} \cdot \frac{dw}{dz}$$

$w = 2z^2 + i$

$\frac{dw}{dz} = 4z$

$\frac{dW}{dw} = 5w^4$

$\frac{dW}{dz} = 5w^4 (4z)$

$= 20zw^4$

$= 20z(2z^2+i)^4$

Derivative liemann Equations:-

Necessary condition for differentiability:-

Theorem:-

A pair of equation that the 1^{st} order partial derivatives of the component function u and v of a function.

$f(z) = u(x,y) + iv(x,y) \rightarrow ①$

must satisfy at any point $z_0 = (x_0, y_0)$ when the derivative at any of f exists there and to express $f'(z_0)$ interms of partial derivatives.

proof:-

Let $f(z) = u(x,y) + iv(x,y)$

W.K.T $\Rightarrow z_0 = x_0 + iy_0$

and $\Delta z = \Delta x + i\Delta y$

Now, $\Delta w = f(z_0 + \Delta z) - f(z_0)$

$= [u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y)] -$
$[u(x_0 + y_0) + iv(x_0, y_0)]$

$= [u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)] + i[v(x_0+\Delta x, y_0, \Delta y)$
$- v(x_0, y_0)] \rightarrow ②$

Assume that the derivative of f exist of z_0,

ie) $f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$

By a known theorem, we know that

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\,\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \rightarrow (3)$$

case: i):

Let us choose the path containing along real axis (horizontaly) through the points $(\Delta x, 0)$

ie) $\Delta x \to 0$, $\Delta y = 0$

(3) $\Rightarrow$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \frac{iv(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

ie)

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \left[\frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta z}\right]$$

$$+ i \lim_{\Delta z \to 0} \left[\frac{v(x_0 + \Delta x, y_0 - v(x_0, y_0)}{\Delta z}\right]$$

$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$

ie) $f'(z_0) = u_x + iv_x \rightarrow (4)$

case: ii)

As it passes through the imaginary axis (vertically) through the points $(0, \Delta y)$

ie) $\Delta x = 0$, $\Delta y \to 0 \rightarrow (1)$

Now (3) $\Rightarrow$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + \frac{iv(x_0, y_0+\Delta y - v(x_0, y_0)}{i\Delta y}$$

$$= \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y}$$

Then

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \left[\frac{-i\,u(x_0, y_0+\Delta y) - u(x_0, y_0)}{\Delta y} \right]$$

$$+ \lim_{\Delta z \to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y}$$

$$= -i \lim_{\Delta z \to 0} \left[\frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{\Delta y} \right] + \lim_{\Delta z \to 0} \left[\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y} \right]$$

$$f'(z_0) = -i\,u_y(x_0, y_0) + v(x_0, y_0)$$

$$= v_y - i u_y \rightarrow (5)$$

From (4) & (5) equating real & imaginary parts, we get

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and}$$

$$v_x(x_0, y_0) = -u_y(x_0, y_0)$$

ie) $u_x = v_y$ and

$u_y = -v_x$

which is the necessary condition for existence of $f'(z_0)$

NOTE :-

(1) ⇒ The equations

$u_x = v_y$ and

$u_y = -v_x$ are called cauchy Riemann Equations (or) C. R Equations.

(2) ⇒ The above results can be summarized as follows.

Suppose that $f(z) = u(x,y) + iv(x,y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. then the 1st order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the CR equations $u_x = v_y$ and $u_y = -v_x$ there also $f'(z_0)$ can be written

$f'(z_0) = u_x + iv_x$ where these partial derivatives are to be evaluated as (x_0, y_0).

Example:-

1. verify C.R. equation for the function $f(z) = z^2$.

Sol:-

Given that, $f(z) = z^2$

$$= (x+iy)^2$$
$$= x^2 - y^2 + 2xiy$$

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

The C.R. equation is

$$u_x = v_y$$
$$u_y = -v_x$$

$$u = x^2 - y^2 \qquad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$2x = 2x$$

$$-2y = -2y$$

C.R is satisfied, so that the function $f(z) = z^2$ is differentiable and

$$f'(z) = 2z$$

2. verify the C.R. equation for the function $f(z) = |z|^2$.

Sol:-

Given that

$$f(z) = |z|^2$$

$$= |x+iy|^2$$

$$= \left(\sqrt{x^2+y^2}\right)^2$$

$$= x^2+y^2$$

$u(x,y) = x^2+y^2$. $v(x,y)=0$

The C.R. equation is

$$u_x = v_y$$

$$u_y = -v_x$$

$u = x^2+y^2$ $v(x,y) = 0$

$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 2y$ $\frac{\partial v}{\partial y} = 0$

$$2x \neq 0$$

$$2y \neq 0$$

C.R is not satisfied so that the function

$f(z) = |z|^2$ is not differentiable.

$f'(z)$ is does not exist

3. verify C.R equation for the function $f(z) = \bar{z}$

Sol:-

Given that

$$f(z) = \bar{z}$$

$$= \overline{(x+iy)}$$

$$f(z) = x-iy$$

$u(x,y) = x$ $v(x,y) = -y$

The C.R equation is

$u_x = v_y$

$u_y = -v_x$

$u(x,y) = x$ $\quad v(x,y) = -y$

$\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\quad \frac{\partial v}{\partial y} = -\phi$

$1 \neq -\phi$

$0 = 0$

C.R is not satisfied so that the function $f(z) = \bar{z}$ is not differentiable

$f'(z)$ is does not exist

④ verify C.R equation for the function $f(z) = z - \bar{z}$

Sol:-

Given that

$f(z) = z - \bar{z}$

$= (x+iy) - (x-iy)$

$f(z) = 2iy$

$u(x,y) = 0$ $\quad v(x,y) = 2y$

$\frac{\partial u}{\partial x} = 0$ $\quad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\quad \frac{\partial v}{\partial y} = 2$

The C.R equation is

$u_x = v_y$

$u_y = -v_x$

$0 \neq 2$

$0 = 0$

C.R is not satisfied so that the function $f(z) = z - \bar{z}$ is not differentiable.

$f'(z)$ is doest exist.

5. verify C.R equation for the function

$f(z) = 2x + ixy^2$

Given that

$f(z) = 2x + ixy^2$

$u(x,y) = 2x \quad , \quad v(x,y) = xy^2$

The C.R equation is

$u_x = v_y$

$u_y = -v_x$

$u(x,y) = 2x \qquad v(x,y) = xy^2$

$\frac{\partial u}{\partial x} = 2 \qquad \frac{\partial v}{\partial x} = y^2$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = y^2 2x$

$2 \neq 2xy$

$0 \neq y^2$

C.R is not satisfied so that the function

$f(z) = 2x + ixy^2$ is not differentiable.

$f'(z)$ is does not exist.

6. verify C.R equation for the function $f(z) = e^x . e^{-iy}$

Given that

$f(z) = e^x . e^{-iy}$

$= e^{x-iy}$

$= e^x (\cos y - i\sin y)$

$= e^x \cos y - ie^x \sin y$

$u(x,y) = e^x \cos y \qquad v(x,y) = -e^x \sin y$

Then C.R equation is

$u_x = v_y$

$u_y = -v_x$

$u(x,y) = e^x \cos y$ $\qquad$ $v(x,y) = -e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y$ $\qquad$ $\frac{\partial v}{\partial x} = -e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$ $\qquad$ $\frac{\partial v}{\partial y} = e^x \cos y$

$$e^x \cos y = e^x \cos y$$

$$-e^x \sin y \neq e^x \sin y$$

C.R is not satisfied so that the function $f(z) = e^x \cdot e^{-iy}$ is not differentiable.

$f'(z)$ is does not exist.

<u>Sufficient condition for differentiability</u>:-

Let the function $f(z) = u(x,y) + iv(x,y)$ be defined throughout some ε-neighborhood of a point $z_0 = x_0 + iy_0$ and suppose that

a) The 1st order partial derivative of the function u and v with respect to x and y exist everywhere in the neighborhood.

b) Those partial derivations are continuous at (x_0, y_0) and satisfy the C.R equations

$u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0)

Then $f'(z_0)$ exist and $f'(z_0) = u_x + iv_x$ at (x_0, y_0)

<u>proof</u>:-

Assume that

i) The 1st order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood.

ii) the partial derivatives are continuous at

(x_0, y_0) and satisfy C.R equations.

$u_x = v_y$ and $u_y = -v_x$

To prove that $f'(z_0)$ exist and $f'(z_0) = u_x + iv_y$

W.K.T $\Rightarrow \Delta z = \Delta x + i\Delta y$ when $0 < |\Delta z| < \varepsilon$

$\Delta w = f(z_0 + \Delta z) - f(z_0)$ and $\Delta w = \Delta u + i\Delta v \rightarrow ①$

ie) $\Delta w = [u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)] + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)]$

where,

$\Delta u = u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) \rightarrow ②$ and

$\Delta v = v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0) \rightarrow ③$

The assumptions that the 1st order partial derivatives of u and v are continuous of a point (x_0, y_0) and by mean valued theorem.

② $\Rightarrow \Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \rightarrow ④$

and

③ $\Rightarrow \Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y \rightarrow ⑤$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$ is Δz plane.

Sub ④ & ⑤ in ①

$\Delta w = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y +$

$i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y] \rightarrow ⑥$

Since, we assume that C.R. eqn's are satisfied,

put $u_x(x_0, y_0) = v_y(x_0, y_0)$ and

$u_y(x_0, y_0) = -v_x(x_0, y_0)$

⑤ $\Rightarrow \Delta w = [u_x(x_0, y_0)\Delta x - v_x(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y] +$

$i[v_x(x_0, y_0)\Delta x + u_x(x_0, y_0)\Delta y) + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y]$

$= u_x(x_0, y_0)\Delta x - v_x(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + i v_x(x_0, y_0)\Delta x$

$+ i u_x(x_0, y_0)\Delta y + i\varepsilon_3 \Delta x + i\varepsilon_4 \Delta y.$

$= u_x(x_0, y_0)(\Delta x + i\Delta y) + i v_x(x_0, y_0)(\Delta x + i\Delta y) + \Delta x(\varepsilon_1 + i\varepsilon_3) +$

$\Delta y(\varepsilon_2 + i\varepsilon_4)$

Divide by Δz

$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + i v_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} + (\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}$$

w.k.T $\Rightarrow |\Delta x| \leq |\Delta z|$ and

$|\Delta y| \leq |\Delta z|$

So, $\left|\frac{\Delta x}{\Delta z}\right| \leq 1$

Consequently,

$$\left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z}\right| \leq |\varepsilon_1 + i\varepsilon_3| \leq |\varepsilon_1| + |\varepsilon_3|$$

and

$$\left|(\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}\right| \leq |\varepsilon_2 + i\varepsilon_4| \leq |\varepsilon_2| + |\varepsilon_4|$$

which means that last two terms tends to zero as $\Delta z \to 0$

⑥ $\Rightarrow \frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + i v_x(x_0, y_0)$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x + i v_x$$

$\Rightarrow f'(z_0) = u_x + i v_x$

$\therefore$ Hence proved.

polar co-ordinates :-

C.R equation is polar co-ordinates :-

Theorem :-

If a function $f(z) = u(r,\theta) + iv(r,\theta)$ and $z = re^{i\theta}$ then C.R equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad (\text{or}) \quad r u_r = v_\theta$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad (\text{or}) \quad r v_r = -u_\theta$$

Proof :-

Let $f(z) = u(r,\theta) + iv(r,\theta)$ and $z = re^{i\theta}$ (or) $z = x+iy$

Assume that $z_0 \neq 0$

put $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

W.K.T $\Rightarrow$ $x = r\cos\theta$ $\quad y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

Now,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$= u_x\cos\theta + u_y\sin\theta \rightarrow ①$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$= -ru_x\sin\theta + u_y$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= v_x\cos\theta + v_y\sin\theta \rightarrow ③$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= -v_x r \sin\theta + v_y r \cos\theta$$

Since $u_x = v_y$ & $u_y = -v_x$

$$\frac{\partial v}{\partial \theta} = -v_x (r \sin\theta) + v_y (r\cos\theta)$$

$$= r[-v_x \sin\theta + v_y \cos\theta]$$

$$= r[u_y \sin\theta + u_x \cos\theta]$$

$$= r[u_x \cos\theta + u_y \sin\theta]$$

$$\frac{\partial v}{\partial \theta} = r \cdot \frac{\partial u}{\partial r} \quad (\text{from } ①)$$

$$\Rightarrow \frac{1}{r} \frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}.$$

② $\rightarrow$ $$\frac{\partial u}{\partial \theta} = -r u_x \sin\theta + u_y r \cos\theta$$

$$= r[-u_x \sin\theta + u_y \cos\theta]$$

$$= r[-v_y \sin\theta - v_x \cos\theta]$$

$$= -r[v_y \sin\theta + v_x \cos\theta]$$

$$= -r \frac{\partial v}{\partial r} \quad (\text{from } ③)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$

Hence proved

① For the function $f(z) = e^z$ if f' exist find $f'(z)$

Sol:-

Given that,

$$f(z) = e^z$$

we know that $f(z) = x + iy$

$$f(z) = e^{x+iy}$$

$$= e^x . e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

$= e^x \cos y + i e^x \sin y$

$u = e^x \cos y \quad , \quad v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$

The C.R equations are

$u_x = v_y$

$u_y = -v_x$

$e^x \cos y = e^x \cos y$

$-e^x \sin y = -e^x \sin y$

Since, C.R equations are satisfied f is differentiable

$\therefore f'$ exist

$f'(z) = u_x + i v_x$

$= e^x \cos y + i e^x \sin y$

$= e^x [\cos y + i \sin y]$

$= e^x e^{iy} = e^{x+iy}$

$= e^z$

$f'(z) = f(z)$

2. $f(z) = |z|^2$

Sol:-

$f(z) = |x+iy|^2$

$= \left(\sqrt{x^2+y^2}\right)^2$

$= x^2+y^2$

$u = x^2+y^2 \quad , \quad v = 0$

C.R equations are not satisfied f is not differentiable hence f' does not exist.

3. $f(z) = \frac{1}{z}$ (solve interms of polar co-ordinates).

Sol :-

$$f(z) = \frac{1}{z}$$

W.K.T $z = re^{i\theta}$

$$f(z) = \frac{1}{re^{i\theta}}$$

$$= \frac{1}{r} e^{-i\theta}$$

$$= \frac{1}{r} (\cos\theta - i\sin\theta)$$

$$= \frac{\cos\theta}{r} - \frac{i\sin\theta}{r}$$

$$u(r,\theta) = \frac{\cos\theta}{r}, \quad v(r,\theta) = -\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial r} = -\cos\theta / r^2, \quad \frac{\partial v}{\partial r} = \frac{\sin\theta}{r^2}$$

$$\frac{\partial u}{\partial \theta} = -\sin\theta / r, \quad \frac{\partial v}{\partial \theta} = \frac{-\cos\theta}{r}$$

C.R. equations interms of polar co-ordinates

are

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial u}{\partial \theta} \text{ and}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

$$\Rightarrow \frac{-\cos\theta}{r^2} = \frac{1}{r} \cdot \frac{-\cos\theta}{r} = \frac{-\cos\theta}{r^2}$$

$$\frac{\sin\theta}{r^2} = \frac{-1}{r} \left(-\frac{\sin\theta}{r} \right)$$

$$\frac{\sin\theta}{r^2} = \frac{\sin\theta}{r^2}$$

$\therefore$ CR equations are satisfied so, f' exist.

NOTE :

i) If $f(z) = u(r,\theta) + iv(r,\theta)$ and it satisfied the C.R. equation $ru_r = v_\theta$ and $-rv_r = u_\theta$

then $f'(z_0) = e^{-i\theta} (u_r + iv_r)$

1) If $f(z) = \frac{1}{z}$ find $f'(z)$

Sol:-

$$f(z) = 1/z$$

$$= \frac{1}{re^{i\theta}}$$

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left(\frac{-\cos\theta}{r^2} + \frac{\sin\theta}{r^2}\right)$$

$$= -e^{-i\theta}\left[\frac{\cos\theta - i\sin\theta}{r^2}\right]$$

$$= -e^{-i\theta}\frac{e^{-i\theta}}{r^2}$$

$$= -\frac{1}{r^2 e^{i\theta} e^{i\theta}} = \frac{-1}{r^2 e^{2i\theta}}$$

$$= -\frac{1}{(re^{i\theta})^2}$$

$$f'(z) = \frac{-1}{z^2}$$

2 for the fun $f(z) = \sqrt[3]{r}\, e^{i\theta/3}$ find f if f' exist

Sol:-

Given, $f(z) = \sqrt[3]{r}\, e^{i\theta/3}$

$$= \sqrt[3]{r}\,(\cos\theta/3 + i\sin\theta/3)$$

$$= r^{1/3}\cos\theta/3 + i r^{1/3}\sin\theta/3$$

Here, $u = r^{1/3}\cos\theta/2$, $v = r^{1/3}\sin\theta/3$

$$\frac{\partial u}{\partial r} = \frac{1}{3} r^{-2/3}\cos\theta/3 \qquad \frac{\partial v}{\partial r} = 1/3\, r^{-2/3}\sin\theta/3$$

$$\frac{\partial u}{\partial \theta} = \frac{-r^{1/3}\sin\theta/3}{3} \qquad \frac{\partial v}{\partial \theta} = \frac{r^{1/3}\cos\theta/3}{3}$$

The C.R equations in polar co-ordinates are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{1}{3} r^{-2/3} \cos\theta/3 = 1/r \; r^{1/3} \frac{\cos\theta/3}{3}$$

$$= r^{-1} r^{1/3} \frac{\cos\theta/3}{3}$$

$$= \frac{r^{-2/3} \cos\theta/3}{3}$$

$$\frac{r^{-2/3} \sin\theta/3}{3} = \frac{-1}{r}\left(-r^{1/3} \frac{\sin\theta/3}{3}\right)$$

$$= \frac{r^{-1} r^{1/3} \sin\theta/3}{3}$$

$$\frac{r^{-2/3} \sin\theta/3}{3} = \frac{r^{-2/3} \sin\theta/3}{3}$$

CR equations are satisfied f' exist

$$f' = e^{-i\theta}(u_r + iv_r)$$

$$= e^{-i\theta}\left[\frac{1}{3(r)^{2/3}} \cos\theta/3 + i \frac{1}{3(-r)^{2/3}} \sin\theta/3\right]$$

$$= e^{-i\theta}\left[\frac{1}{3(r)^{2/3}} (\cos\theta/3 + i\sin\theta/3)\right]$$

$$= e^{-i\theta}\left[\frac{1}{3(r)^{2/3}} e^{i\theta/3}\right]$$

$$= \frac{1}{3(\sqrt[3]{r})^2}\left\{\frac{1}{e^{i\theta} \cdot e^{-i\theta/3}}\right\}$$

$$= \frac{1}{3(\sqrt[3]{r})^2} \cdot \frac{1}{e^{2i\theta/3}}$$

$$= \frac{1}{3(\sqrt[3]{r})^2} \cdot \frac{1}{(e^{i\theta/3})^2}$$

$$f' = \frac{1}{3(\sqrt[3]{r} \cdot e^{i\theta/3})^2}$$

$$f' = \frac{1}{3f(z)}$$

Analytic function / Regular / holomorphic :-

A function $f(z)$ of a complex variable z is analytic at a point z_0 if it has a derivative at each point in some neighborhood of z_0.

(i) if f is analytic at a point z_0, it must to be analytic at each point in some neighborhood of z_0.

NOTE :-

i) $f(z) = 1/z$ is analytic at each non-zero point in a finite plane.

ii) $f(z) = |z|^2$ is not analytic since derivatives exist only at $z=0$ and not throughout the neighborhood.

Entire function :-

An entire function is a function that is analytic at each point in the entire finite plane.

NOTE :-

Since derivative of a polynomial exist everywhere, every polynomial is a entire function.

Singular point:

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a singular point of f (or) singularity.

Eg:-

1. For the function $f(z) = 1/z$; $z=0$ is a singular point.

2. $f(z) = |z|^2$ has no ~~singular~~ singular point.

Remark :-

① If two functions are analytic in a domain D their sum & product are both analytic in D.

② Their quotient is analytic provides that the function of denominator doesnot vanish.

ie) $P(z) / Q(z)$ is analytic, when $Q(z) \neq 0$.

③ composition of 2 analytic function is analytic.

Theorem :-

If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.

Sol :-

Let $f(z) = u(x, y) + iv(x, y)$

assume that $f'(z) = 0$ everywhere in D.

To prove that : $f(z)$ is constant in D.

WKT $\Rightarrow f'(z) = u_x + iv_x$ (or)

$f'(z) = u_y + iv_y$

given that $f'(z) = 0$

$\Rightarrow u_x + iv_x = 0$.

$u_x(x, y) + iv_x(x, y) = 0 + i0$

$u_x(x, y) = 0$, $v_x(x, y) = 0$

$\Rightarrow u_y + iv_y = 0 + i0$

Equating real & imaginary part

$u_y(x, y) = 0$, $v_y(x, y) = 0$

By the C.R eqn,

$u_x = v_y$ & $u_y = -v_x$

$\Rightarrow u_x = v_y = u_y = -v_x = 0$

$u(x, y)$ & $v(x, y)$ are constant

$\therefore f(z) = u+iv$ is constant throughout D.

Hence proved.

Next consider $u(x,y)$ is constant as a function of real variables.

Let p be a point in D and let p' be another point in D which lies on a line L which lies in D.

Let υ denotes the unit vector along the line L directed from p to p'.

Let s denote the distance along L from p.

The directional derivative of $u(x,y)$ and along line L is

$$\frac{du}{ds} = \text{grad}(u) \cdot \upsilon \text{ where } \rightarrow ①$$

$$\text{grad } u = \Delta u = u_x(x,y)i + v_y(x,y)j$$

(ie) $\Delta u = u_x i + v_y j \rightarrow ②$

Because u_x and u_y are non-zero every where in D grad u is evidently zero vector at all points on L

Since $u_x(x,y) = u_y(x,y) = 0 \ \forall \ x,y \in D$

Then grad $(u) = 0$ at all point along L so u is a constant on L and value of u at point P is same as its value of p'

$\therefore f(z)$ is constant throughout D.

Examples:-

① $\rightarrow$ The quotient $f(z) = \dfrac{z^3+4}{(z^2-3)(z^2+1)}$ is evidenty analytic throughout the z plane expect for singular points

$z = \pm\sqrt{3}, \quad z = \pm i$

2. verify the function $f(z) = \cosh x \cos y + i \sinh x \sin y$ is analytic.

Sol:-

Given that

$$f(z) = \cosh x \cdot \cos y + i \sinh x \sin y$$

Here,

$$u(x,y) = \cosh x \cos y$$

$$\frac{\partial u}{\partial x} = \sinh x \cos y$$

$$\frac{\partial u}{\partial y} = \cosh x - \sin y$$

$$v(x,y) = \sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \cosh x \sin y$$

$$\frac{\partial v}{\partial y} = \sinh x \cos y$$

So C.R equations are $u_x = v_y$

$u_y = -v_x$

$$\sinh x \cos y = \sinh x \cos y$$

$$\cosh x - \sin y = -\cosh x \sin y$$

C.R equations are satisfied.

$\therefore f(z)$ is analytic function.

Theorem:-

Suppose that the function $f(z) = u(x,y) + iv(x,y)$ and its conjugate $\overline{f(z)}$ are both analytic in D show that $f(z)$ must be constant throughout D.

proof:-

Let $f(z) = u(x,y) + iv(x,y)$ and

To prove that $f(z)$ must be constant throughout D.

Now,

$\overline{f(z)} = u(x,y) - iv(x,y)$

Let $F(z) = U(x,y) + iV(x,y)$

where, $U(x,y) = u(x,y)$ and $V(x,y) = -v(x,y)$ → ①

Since, $f(z)$ is analytic, it satisfies the C.R equation

$$u_x = v_y \; \& \; u_y = -v_x \rightarrow ②$$

Since $\overline{f(z)}$ is also analytic, it also satisfies the C.R. equation

$$U_x = -V_y \text{ and } U_y = V_x \rightarrow ③$$

from ① and ②

$$u_x = v_y \text{ and } u_y = -v_x \rightarrow ④$$

③ + ④ ⇒ $u_x = v_y - v_y = 0$

$u_y = -v_x + v_x = 0$

$\Rightarrow u_x = u_y = 0$

$\therefore f'(z) = u_x + iv_x = 0 + i0 = 0$

$f(z)$ is constant throughout D.

Theorem :-

If $|f(z)|$ is constant in a domain D has be constant then D prove that $f(z)$ is constant in D.

proof :-

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic in D.

Since, $|f(z)|$ is constant

Then $|f(z)| = c$

$\Rightarrow \sqrt{u^2+v^2} = c$

$u^2 + v^2 = c$

Let $c^2 = k$

$k = u^2 + v^2 \rightarrow ①$

Diff ① p.w.r to "x"

$① \Rightarrow 0 = 2u u_x + 2v v_x$

$= 2[u u_x + v v_x]$

$u u_x + v v_x = 0 \rightarrow ②$

Diff ① p.w.r to "y"

$0 = 2u u_y + 2v v_y$

$= 2[u u_y + v v_y]$

$u u_y + v v_y = 0 \rightarrow ③$

Since, $f(z)$ is analytic is satisfies C.R equation

$u_x = v_y$ and $u_y = -v_x$

$② \Rightarrow u u_x - v u_y = 0 \rightarrow ④$

$③ \Rightarrow u u_y + v u_x = 0 \rightarrow ⑤$

$④ \Rightarrow u u_x = v u_y$

$u_x = \frac{v}{u} u_y$

$⑤ \Rightarrow u u_y = -v u_x$

$u_x = -\frac{u}{v} u_y$

$\Rightarrow \frac{v}{u} u_y = -\frac{u}{v} u_y$

$v^2 = -u^2$

$u^2 + v^2 = 0$

$u^2 = 0, \; v^2 = 0$

$u = 0, v = 0$

$\Rightarrow u(x,y)$ & $v(x,y)$ are constant.

$\therefore f(z) = u + iv$ are constant.

Harmonic function :-

A real valued function H of two real variables x and y is said to be harmonic in a given domain of xy plane if throughout that domain, it has continuous partial derivatives of first and second order and satisfies the partial differential equation

$$H_{xx}(x,y) + H_{yy}(x,y) = 0$$

The above equation is called Laplace equation.

Theorem:-

If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.

proof:-

Given that,

$f(z) = u(x,y) + iv(x,y)$ is analytic in D.

The function u and v are harmonic in D.

If a function of complex variable is analytic at a point, then its real and Imaginary components have a continuous partial derivatives of all orders.

Since, $f(z)$ is analytic is satisfies the CR equation

$$u_x = v_y$$

$$u_y = -v_x$$

Diff w.r to "x"

$$u_{xx} = v_{yx} \; ; \quad u_{yx} = -v_{xx}$$

Diff w.r to "y"

$$u_{yx} = u_{xy} \text{ and } v_{xy} = v_{yx}$$

$$u_{xx} + u_{yy} = 0$$

$$v_{xx} + v_{yy} = 0$$

u and v are harmonic in D.

Conjugate harmonic:-

If $f(z) = u(x,y) + iv(x,y)$ is an analytic function Then v is said to be harmonic conjugate of u.

Theorem:-

A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is harmonic conjugate of u.

Proof:

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function. Then if v is harmonic conjugate u. Then first order partial derivative satisfies C.R equation.

So f is differentiable and analytic in D

Conversely assume that, f(z) is analytic in D, then w.k.T u and v are harmonic in D By the definition of analytic f is differentiable throughout D and u are v satisfies C.R. equation. Therefore v is harmonic conjugate of u.

∴ Hence proved.

Example:

1. $u = y^3 - 3x^2y$ find the harmonic conjugate v.

Sol:-

Let $u = y^3 - 3x^2y$ be a harmonic throughout D. Since a harmonic conjugate of $v(x,y)$ is related to $u(x,y)$

It satisfies C.R. eqn,

$$u(x,y) = y^3 - 3x^2y$$

$$u_x = -6xy \qquad u_y = 3y^2 - 3x^2$$

$$u_{xx} = -6y \qquad u_{yy} = 6y$$

Now $u_x = v_y = -6xy$

Sing w.r to "y"

$$\int v_y\, dy = \int -6xy\, dy$$

$v = -3xy^2 + \phi(x)$ where ϕ is a arbitary fun of x.

$u_y = -v_x$

$\Rightarrow -3y^2 - \phi'(x) = 3y^2 - 3x^2$

$\Rightarrow \phi'(x) = 3x^2$

$\phi = \cancel{x}^3/3 + c$ where c is constant

$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3) + c$

(1) $\Rightarrow v = -3xy^2 + x^3 + c$ is harmonic and conjugate of u.