17UMAE03

DISCRETE MATHEMATICS

III B.SC MATHEMATICS

V SEMESTER

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UNIT - II

Normal forms - Disjunctive Normal forms - Conjunctive Normal forms - Principal Disjunctive Normal forms - Principal conjunctive Normal forms - Ordering and uniqueness of Normal forms The theory of inference for the statement calculus - Validity using truth tables - Rules of inference.
(sections 1.3.1 to 1.3.5, 1.4.1 to 1.4.2)

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Normal forms - Disjunctive Normal forms - Conjunctive Normal forms - principal Disjunctive Normal forms - principal Conjunctive Normal forms - ordering and uniqueness of normal forms the theory of inference for the statement calculus - validity using truth tables - Rules of inference. (sections 1.3.1 to 1.3.5, 1.4.1 to 1.4.2)

UNIT - II

Normal Forms

Satisfiable:

A compound proposition $A(P_1, P_2, P_3 \dots P_n)$ is said to be satisfiable if it has a truth value TRUE for atleast one combination of the truth values of $P_1, P_2, \dots P_n$.

Types of Normal forms:

1. Disjunctive Normal forms (DNF)
2. Conjunctive Normal forms (CNF)

Elementary product:

A Product of the variables and their negation in a formula is called an elementary product

Ex: P, $P \wedge Q$, $P \wedge \neg P$, $\neg P \wedge Q$, $P \wedge \neg P \wedge Q$, $\neg P \wedge \neg Q$

Elementary Sum:

A Sum of the variables and their negations in a formula is called an elementary sum.

Ex: P, $P \vee \neg Q$, $P \vee \neg P$, $\neg P \vee Q$, $P \vee \neg P \vee Q$, $\neg P \vee \neg Q$

Disjunctive Normal Forms: (DNF)

A compound proposition (or) a formula which consists of a sum of elementary products and which is equivalent to the given proposition is called a DNF.

Ex: $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Conjunctive Normal Forms : (CNF)

A compound proposition (or) a formula which consists of a product of elementary sums and which is equivalent to the given proposition is called a CNF.

Ex: $(\neg P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg Q \vee \neg P)$

Find the DNF of the following:-

1. $P \wedge (P \to Q)$

soln:

$$P \wedge (P \to Q) \Leftrightarrow P \wedge (\neg P \vee Q)$$
$$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$$

which is the required DNF.

2. $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$$\neg(P \vee Q) \rightleftarrows (P \wedge Q) \Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee (\neg(\neg(P \vee Q)) \wedge \neg(P \wedge Q))$$
$$\Leftrightarrow (\neg P \wedge \neg Q) \wedge (P \wedge Q) \vee (\neg(\neg P \wedge \neg Q)) \wedge (\neg P \vee \neg Q)$$
$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$$
$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$$
$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$$

which is the required DNF.

3. $\neg(\neg(P \rightleftarrows Q) \wedge R)$

$$\Leftrightarrow \neg(\neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \wedge R)$$
$$\Leftrightarrow \neg((\neg(P \wedge Q) \wedge \neg(\neg P \wedge \neg Q) \wedge R)$$
$$\Leftrightarrow \neg(((\neg P \vee \neg Q) \wedge (P \vee Q)) \wedge R)$$
$$\Leftrightarrow \neg(((\neg P \wedge (P \vee Q)) \vee (\neg Q \wedge (P \vee Q))) \wedge R)$$
$$\Leftrightarrow \neg(((\neg P \wedge P) \vee (\neg P \wedge Q) \vee (\neg Q \wedge P) \vee (\neg Q \wedge Q)) \wedge R)$$
$$\Leftrightarrow \neg(((F \vee (\neg P \wedge Q) \vee (\neg Q \wedge P) \vee F) \wedge R)$$
$$\Leftrightarrow \neg(((\neg P \wedge Q) \vee (\neg Q \wedge P)) \wedge R)$$
$$\Leftrightarrow \neg(((\neg P \vee (\neg Q \wedge P)) \wedge (Q \vee (\neg Q \wedge P))) \wedge R)$$
$$\Leftrightarrow \neg(((\neg P \vee \neg Q) \wedge (\neg P \vee P) \wedge (Q \vee \neg Q) \wedge (Q \vee P)) \wedge R)$$
$$\Leftrightarrow \neg(((\neg P \vee \neg Q) \wedge (Q \vee P)) \wedge R)$$
$$\Leftrightarrow \neg(\neg P \vee \neg Q) \vee \neg(Q \vee P) \vee \neg R$$
$$\Leftrightarrow (P \wedge Q) \vee (\neg Q \wedge \neg P) \vee \neg R$$

which is the required DNF.

4. $P \vee (\neg P \to (Q \vee (Q \to \neg R)))$

$\Leftrightarrow P \vee (\neg P \to (Q \vee (\neg Q \vee \neg R)))$

$\Leftrightarrow P \vee (P \vee (Q \vee \neg Q \vee \neg R)))$

$\Leftrightarrow P \vee P \vee Q \vee \neg Q \vee \neg R$

$\Leftrightarrow P \vee Q \vee \neg Q \vee \neg R$

which is the required DNF.

5. $(P \wedge \neg (Q \wedge R)) \vee (P \to Q)$

$\Leftrightarrow (P \wedge \neg (Q \wedge R)) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \wedge (\neg Q \vee \neg R)) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg Q) \vee (P \wedge \neg R) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg Q) \vee (P \wedge \neg R) \vee \neg P \vee Q$

which is the required DNF.

6. $(P \wedge \neg (Q \vee R)) \vee (((P \wedge Q) \vee \neg R) \wedge P)$

$\Leftrightarrow (P \wedge (\neg Q \wedge \neg R)) \vee ((P \wedge Q) \wedge P) \vee (\neg R \wedge P)$

$\Leftrightarrow (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge P) \vee (P \wedge \neg R)$

$\Leftrightarrow (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q) \vee (P \wedge \neg R)$

which is the required DNF.

Find the CNF of the following.

1. $(P \wedge \neg (Q \wedge R)) \vee (P \to Q)$

$\Leftrightarrow (P \wedge (\neg Q \vee \neg R)) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg Q) \vee (P \wedge \neg R) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \vee (P \wedge \neg R)) \wedge (\neg Q \vee (P \wedge \neg R)) \vee (\neg P \vee Q)$

$\Leftrightarrow P \wedge (\neg Q \vee P) \wedge (\neg Q \vee \neg R) \vee (\neg P \vee Q)$

$\Leftrightarrow P \wedge (\neg Q \vee Q \vee \neg R \vee \neg P)$

$\Leftrightarrow P \wedge (T \vee (\neg R \vee \neg P)$

$\Leftrightarrow P \wedge T$

$\Leftrightarrow P$

which is the required CNF.

2. $(Q \vee (P \wedge Q)) \wedge \neg ((P \vee R) \wedge Q)$

$\Leftrightarrow (Q \vee (Q \wedge P)) \wedge \neg ((P \vee R) \wedge Q)$

$\Leftrightarrow Q \wedge (\neg P \wedge \neg R) \vee \neg Q$

$\Leftrightarrow Q \wedge (\neg Q \vee (\neg P \wedge \neg R)$

$\Leftrightarrow Q \wedge (\neg Q \vee \neg P) \wedge (\neg Q \vee \neg R)$

which is the required CNF.

Minterms:

Given a number of variables, the products (or $\wedge$) in which each variables (or) its negation but not both occurs only one are called minterms.

For two variables P and Q the possible minterms are

$m_0 = m_{00} = \neg P \wedge \neg Q$
$m_1 = m_{01} = \neg P \wedge Q$
$m_2 = m_{10} = P \wedge \neg Q$
$m_3 = m_{11} = P \wedge Q$

For 3 variables P, Q and R the possible minterms are

$m_0 = m_{000} = \neg P \wedge \neg Q \wedge \neg R$
$m_1 = m_{001} = \neg P \wedge \neg Q \wedge R$
$m_2 = m_{010} = \neg P \wedge Q \wedge \neg R$
$m_3 = m_{011} = \neg P \wedge Q \wedge R$
$m_4 = m_{100} = P \wedge \neg R \wedge \neg Q$
$m_5 = m_{101} = P \wedge \neg Q \wedge R$
$m_7 = m_{111} = P \wedge Q \wedge R$

Maxterms:

Given a number of variables, the sums (or $\vee$) in which each variables (or) its negation but not both occurs only once are called maxterms.

For two variables P and Q the possible maxterms are

$M_0 = M_{00} = P \vee Q$
$M_1 = M_{01} = P \vee \neg Q$
$M_2 = M_{10} = \neg P \vee Q$
$M_3 = M_{11} = \neg P \vee \neg Q$

For 3 variables P, Q and R, the possible maxterms are

$M_0 = M_{000} = P \vee Q \vee R$
$M_1 = M_{001} = P \vee Q \vee \neg R$
$M_2 = M_{010} = P \vee \neg Q \vee R$
$M_3 = M_{011} = P \vee \neg Q \vee \neg R$
$M_4 = M_{100} = \neg P \vee Q \vee R$
$M_5 = M_{101} = \neg P \vee Q \vee \neg R$

$M_6 = M_{110} = \neg P \vee \neg Q \vee R$

$M_7 = M_{111} = \neg P \vee \neg Q \vee \neg R$

Principal Disjunctive Normal Forms (PDNF)

(or)

Sum-of products canonical form

A formula (compound proposition) consisting of disjunction of minterms in variables only and equivalent to the given formula is known as PDNF (or) sum-of-Products canonical form.

Principal Conjunctive Normal Forms (PCNF)

(or)

Product-of-sums canonical form

A formula (compound proposition) consisting of conjunctions of maxterms in variables only and equivalent to the given formula is known as PCNF (or) Product of sums canonical form.

Note: 1

If the given formula is a tautology then its PDNF includes all the possible terms of the variables and there is no PCNF.

Note: 2

If the given formula is a contradiction then its PCNF includes all the possible maxterms of the variables and there is no PDNF.

1. Obtain the PDNF and PCNF of the following formula using truth tables.

(i) $(\neg P \vee \neg Q) \rightarrow (P \rightleftarrows \neg Q)$

(ii) $P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R)))$

(iii) $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg P \wedge \neg R))$.

1. Let $S: (\neg P \vee \neg Q) \rightarrow (P \rightleftarrows \neg Q)$

P	Q	$\neg P$	$\neg Q$	$(\neg P \vee \neg Q)$	$(P \rightleftarrows \neg Q)$	$(\neg P \vee \neg Q) \rightarrow (P \rightleftarrows \neg Q)$
T	T	F	F	F	F	T
T	F	F	T	T	T	F
F	T	T	F	T	T	T
F	F	T	T	T	F	F

PDNF of $S \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

PCNF of $S \Leftrightarrow (P \vee Q)$

2. Let $S: P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R)))$

P	Q	R	$\neg P$	$\neg Q$	$\neg Q \rightarrow R$	$Q \vee (\neg Q \rightarrow R)$	$\neg P \rightarrow (Q \vee (\neg Q \rightarrow R))$	$P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R)))$
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	F	T	T
F	T	T	T	F	T	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	F	F

PDNF of $S \Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee$
$(\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q)$

PCNF of $S \Leftrightarrow (P \vee Q \vee R)$

3. Let $S: (P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg P \wedge \neg R))$.

P	Q	R	$Q \wedge R$	$P \rightarrow (Q \wedge R)$	$\neg P$	$\neg R$	$\neg P \wedge \neg R$	$\neg P \rightarrow (\neg P \wedge \neg R)$	$P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg P \wedge \neg R))$
T	T	T	T	T	F	F	F	T	T
T	T	F	F	F	F	T	F	T	F
T	F	T	F	F	F	F	F	T	F
T	F	F	F	F	F	T	F	T	F
F	T	T	T	T	T	F	F	F	F
F	T	F	F	T	T	T	T	T	T
F	F	T	F	T	T	F	F	F	F
F	F	F	F	T	T	T	T	T	T

PDNF of $S \Leftrightarrow (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

PCNF of $S \Leftrightarrow (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge$
$(P \vee Q \vee \neg R)$

4. S: $(\neg P \to R) \wedge (Q \rightleftarrows P)$

P	Q	R	$\neg P$	$\neg P \to R$	$Q \rightleftarrows P$	$(\neg P \to R) \wedge (Q \rightleftarrows P)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

PDNF of $S \Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R)$

PCNF of $S \Leftrightarrow (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R)$

Without constructing truth table, find the PDNF and the PCNF of the following compound proposition.

1. $(P \wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$

Let $S: (P \wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$

$\Leftrightarrow ((P \wedge Q) \wedge T) \vee ((\neg P \wedge Q) \wedge T) \vee ((Q \wedge R) \wedge T)$

$\Leftrightarrow ((P \wedge Q) \wedge (R \vee \neg R)) \vee ((\neg P \wedge Q) \wedge (R \vee \neg R)) \wedge ((Q \wedge R) \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$

$\Leftrightarrow m_{111} \vee m_{110} \vee m_{011} \vee m_{010}$

$\Leftrightarrow m_7 \vee m_6 \vee m_3 \vee m_2$

$S \Leftrightarrow \sum_{2,3,6,7}$ which is the PDNF of S

PDNF of $\neg S \Leftrightarrow \sum_{0,1,4,5}$

$\Leftrightarrow m_0 \vee m_1 \vee m_4 \vee m_5$

$\Leftrightarrow m_{000} \vee m_{001} \vee m_{100} \vee m_{101}$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$

PCNF of $S \Leftrightarrow \neg(\text{PDNF of } \neg S)$

$\therefore$ PCNF of $S \Leftrightarrow (P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\Leftrightarrow M_{000} \wedge M_{001} \wedge M_{100} \wedge M_{101}$

$\Leftrightarrow M_0 \wedge M_1 \wedge M_4 \wedge M_5$

$\Leftrightarrow \prod_{0,1,4,5}$ which is the PCNF of S.

2. $(\neg P \to Q) \wedge (Q \rightleftarrows P)$

Let S: $(\neg P \to Q) \wedge (Q \rightleftarrows P)$

$\Leftrightarrow (\neg\neg P \vee Q) \wedge ((Q \wedge P) \vee (\neg Q \wedge \neg P))$

$\Leftrightarrow (P \vee Q) \wedge ((P \wedge Q) \vee (\neg Q \wedge \neg P)$

$\Leftrightarrow ((P \vee Q) \wedge (P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \wedge \neg Q))$

$\Leftrightarrow ((P \wedge P \wedge Q) \vee (Q \wedge P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \vee Q)$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge Q) \vee \mathbb{F}$

$\Leftrightarrow (P \wedge Q) \vee \mathbb{F}$

$\Leftrightarrow P \wedge Q$

$\Leftrightarrow m_{11}$

$\Leftrightarrow m_3$

$S \Leftrightarrow \sum_3$ which is the PDNF of S.

PDNF of $\neg S \Leftrightarrow \sum_{0,1,2}$

$\Leftrightarrow m_0 \vee m_1 \vee m_2$

$\Leftrightarrow m_{00} \vee m_{01} \vee m_{10}$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q)$

$\therefore$ PCNF of $S \Leftrightarrow \neg$ (PDNF of $\neg S$)

$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q)$

$\Leftrightarrow M_{00} \wedge M_{01} \wedge M_{10}$

$\Leftrightarrow M_0 \wedge M_1 \wedge M_2$

$\Leftrightarrow \prod_{0,1,2}$ which is the PCNF of S.

3. $(P \wedge \neg(Q \wedge R)) \vee (P \to Q)$

Let S: $(P \wedge \neg(Q \wedge R)) \vee (P \to Q)$

$\Leftrightarrow (P \wedge (\neg Q \vee \neg R)) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg Q) \vee (P \wedge \neg R) \vee (\neg P \vee Q)$

$\Leftrightarrow ((P \wedge \neg Q) \wedge \mathbb{T}) \vee ((P \wedge \neg R) \wedge \mathbb{T} \vee ((\neg P \vee Q) \wedge \mathbb{T}$

$\Leftrightarrow ((P \wedge \neg Q) \wedge (R \vee \neg R)) \vee ((P \wedge \neg R) \wedge (Q \vee \neg Q)) \vee$
$((\neg P \vee Q) \wedge (R \vee \neg R)$

$\Leftrightarrow (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$
$\vee ((\neg P \wedge (R \vee \neg R) \vee ((Q \wedge (R \vee \neg R)$

$\Leftrightarrow (P\wedge\neg Q\wedge R)\vee(P\wedge\neg Q\wedge\neg R)\vee(P\wedge Q\wedge\neg R)\vee(P\wedge\neg Q\wedge\neg R)\vee(\neg P\wedge R)\vee$
$(\neg P\wedge\neg R)\vee(Q\wedge R)\vee(Q\wedge\neg R)$

$\Leftrightarrow (P\wedge\neg Q\wedge R)\vee(P\wedge\neg Q\wedge\neg R)\vee(P\wedge Q\wedge\neg R)\vee(P\wedge\neg Q\wedge\neg R)\vee((\neg P\wedge R)\wedge$
$(Q\vee\neg Q))\vee((\neg P\wedge\neg R)\wedge(Q\vee\neg Q))\vee((Q\wedge R)\wedge(P\vee\neg P)\vee$
$((Q\wedge\neg R)\wedge(P\vee\neg P)$

$\Leftrightarrow (P\wedge\neg Q\wedge R)\vee(P\wedge\neg Q\wedge\neg R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge R)\vee$
$(\neg P\wedge Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge\neg R)\vee(P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge R)\vee$
$(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge\neg R)$

$\Leftrightarrow (P\wedge\neg Q\wedge R)\vee(P\wedge\neg Q\wedge\neg R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge R)\vee$
$(\neg P\wedge Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge\neg R)\vee(P\wedge Q\wedge R)$

$\Leftrightarrow m_{101}\vee m_{100}\vee m_{110}\vee m_{001}\vee m_{010}\vee m_{000}\vee m_{111}$

$\Leftrightarrow m_5\vee m_4\vee m_6\vee m_3\vee m_1\vee m_2\vee m_0\vee m_7$

$\Leftrightarrow \Sigma_{0,1,2,3,4,5,6,7}$ which is the required PDNF of S.

4. $(Q\vee(P\wedge R))\wedge\neg((P\vee R)\wedge Q)$

$\Leftrightarrow (Q\vee P)\wedge(Q\vee R)\wedge((\neg P\wedge\neg R)\vee\neg Q)$

$\Leftrightarrow (P\vee Q)\wedge(Q\wedge R)\wedge(\neg P\vee\neg Q)\wedge(\neg R\vee\neg Q)$

$\Leftrightarrow ((P\vee Q)\vee F)\wedge((Q\vee R\vee F)\wedge((\neg P\vee\neg Q\vee F)\wedge((\neg Q\vee\neg R\vee F)$

$\Leftrightarrow ((P\vee Q)\vee(R\wedge\neg R))\wedge((Q\vee R)\vee(P\wedge\neg P))\wedge((\neg P\vee\neg Q)\vee(R\wedge\neg R)\wedge$
$((\neg Q\vee\neg R)\vee(P\wedge\neg P))$

$\Leftrightarrow (P\vee Q\vee R)\wedge(P\vee Q\vee\neg R)\wedge(P\vee Q\vee R)\wedge(\neg P\vee Q\vee R)\wedge(\neg P\wedge\neg Q\vee R)\wedge$
$(\neg P\vee\neg Q\vee\neg R)\wedge(P\vee\neg Q\vee\neg R)\wedge$
$(\neg P\vee\neg Q\vee\neg R)$

$\Leftrightarrow (P\vee Q\vee R)\wedge(P\vee Q\vee\neg R)\wedge(\neg P\vee Q\vee R)\wedge(\neg P\vee\neg Q\vee R)\wedge(\neg P\vee\neg Q\vee\neg R)\wedge$
$(P\vee\neg Q\vee\neg R)$

$\Leftrightarrow M_{000}\wedge M_{001}\wedge M_{100}\wedge M_{110}\wedge M_{111}\wedge M_{011}$

$\Leftrightarrow M_0\wedge M_1\wedge M_4\wedge M_6\wedge M_7\wedge M_3$

$\Leftrightarrow \Pi_{0,1,3,4,6,7}$ which is the required PCNF of S.

PCNF of $\neg S \Leftrightarrow \Pi_{2,5}$

$\Leftrightarrow M_2\wedge M_5$

$\Leftrightarrow M_{010}\wedge M_{101}$

$\Leftrightarrow (P\vee\neg Q\vee R)\wedge(\neg P\vee Q\vee\neg R)$

$\therefore$ PDNF of S $\Leftrightarrow \neg$(PCNF of $\neg S$)

$\Leftrightarrow (\neg P\wedge Q\wedge\neg R)\vee(P\wedge\neg Q\wedge R)$

$\Leftrightarrow m_{010} \vee m_{101}$

$\Leftrightarrow m_2 \vee m_5$

$\Leftrightarrow \Sigma_{2,5}$ which is the required PDNF of S.

5. $(\neg P \rightarrow R) \wedge (Q \rightleftarrows P)$

Let $S \Leftrightarrow (\neg\neg P \vee R) \wedge (Q \rightarrow P) \wedge (P \rightarrow Q)$

$\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$

$\Leftrightarrow ((P \vee R) \vee F) \wedge ((\neg Q \vee P) \vee F) \wedge ((\neg P \vee Q) \vee F)$

$\Leftrightarrow ((P \vee R) \vee (Q \wedge \neg Q)) \wedge ((\neg Q \vee P) \vee (R \wedge \neg R)) \wedge ((\neg P \vee Q) \vee (R \wedge \neg R))$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\Leftrightarrow M_{000} \wedge M_{010} \wedge M_{011} \wedge M_{100} \wedge M_{101}$

$\Leftrightarrow M_0 \wedge M_2 \wedge M_3 \wedge M_4 \wedge M_5$

$S \Leftrightarrow \Pi_{0,2,3,4,5}$ which is the required PCNF of S.

PCNF of $\neg S \Leftrightarrow \Pi_{1,6,7}$

$\Leftrightarrow M_1 \wedge M_6 \wedge M_7$

$\Leftrightarrow M_{001} \wedge M_{110} \wedge M_{111}$

$\Leftrightarrow (P \vee Q \vee \neg R)(\neg P \vee \neg Q \vee R) \wedge (\neg P \wedge \neg Q \wedge \neg R)$

$\therefore$ PDNF of S $\Leftrightarrow \neg$(PCNF of $\neg S$)

$\Leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

$\Leftrightarrow m_{001} \vee m_{110} \vee m_{111}$

$\Leftrightarrow m_1 \vee m_6 \vee m_7$

$\Leftrightarrow \Sigma_{1,6,7}$ which is the required PDNF of S.

6. $S: (P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$

$S \Leftrightarrow (\neg P \vee (Q \wedge R)) \wedge (\neg\neg P \vee (\neg Q \wedge \neg R))$

$\Leftrightarrow (\neg P \vee (Q \wedge R)) \wedge (P \vee (\neg Q \wedge \neg R))$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee R) \wedge (P \vee \neg Q) \wedge (P \vee \neg R)$

$\Leftrightarrow ((\neg P \vee Q) \vee F) \wedge ((\neg P \vee R) \vee F) \wedge ((P \vee \neg Q) \vee F) \wedge ((P \vee \neg R \vee F)$

$\Leftrightarrow ((\neg P \vee Q) \vee (R \wedge \neg R)) \wedge ((\neg P \vee R) \vee (Q \wedge \neg Q)) \wedge ((P \vee \neg Q) \vee (R \wedge \neg R)) \wedge ((P \vee \neg R) \vee (Q \wedge \neg Q))$

$\Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$

$\Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge$
$(P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R)$

$\Leftrightarrow M_{100} \wedge M_{101} \wedge M_{110} \wedge M_{010} \wedge M_{011} \wedge M_{001}$

$\Leftrightarrow M_4 \wedge M_5 \wedge M_6 \wedge M_2 \wedge M_3 \wedge M_1$

$S \Leftrightarrow \Pi_{1,2,3,4,5,6}$ which is the required PCNF of S.

PCNF of $\neg S \Leftrightarrow \Pi_{0,7}$

$\Leftrightarrow M_0 \wedge M_7$

$\Leftrightarrow M_{000} \wedge M_{111}$

$\Leftrightarrow (P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

$\therefore$ PDNF of $S \Leftrightarrow \neg$(PCNF of $\neg S$)

$\Leftrightarrow (\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

$\Leftrightarrow m_{000} \vee m_{111}$

$\Leftrightarrow m_0 \vee m_7$

$\Leftrightarrow \Sigma_{0,7}$ which is the required PDNF of S.

The theory of inference for statement calculus:

Premise (or) Hypothesis:

A Premise is a statement which is assumed to be true.

Theorem:

A theorem consists of a set of premise and a conclusion. A theorem is proved by showing that the conclusion in true whenever all the premises are assumed to be true.

Formal Proof:

The process of determing a conclusion from a set of premises by using the accepted rules of reasoning is called a formal proof.

Types of Proofs:

1. Direct proof
2. Indirect proof (or) proof by contradiction.
3. conditional proof (or) proof by deduction.

Implications:

$$\left.\begin{array}{l} I_1 : P \wedge Q \Rightarrow P \\ I_2 : P \wedge Q \Rightarrow Q \end{array}\right\} \text{simplification}$$

$I_3 : P \Rightarrow P \vee Q$
$I_4 : Q \Rightarrow P \vee Q$ } addition
$I_5 : \neg P \Rightarrow P \rightarrow Q$
$I_6 : \neg Q \Rightarrow P \rightarrow Q$
$I_7 : \neg(P \rightarrow Q) \Rightarrow P$
$I_8 : \neg(P \rightarrow Q) \Rightarrow \neg Q$
$I_9 : P, Q \Rightarrow P \wedge Q$
$I_{10} : \neg P, P \vee Q \Rightarrow Q$ (disjunctive syllogism)
$I_{11} : P, P \rightarrow Q \Rightarrow Q$ (Modus Ponens)
$I_{12} : \neg Q, P \rightarrow Q \Rightarrow \neg P$ (Modus tollens)
$I_{13} : P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (Hypothetical syllogism)
$I_{14} : P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (dilemma)

Equivalences:

$E_1 : \neg\neg P \Leftrightarrow P$ (Double Negative)
$E_2 : P \wedge Q \Leftrightarrow Q \wedge P$
$E_3 : P \vee Q \Leftrightarrow Q \vee P$ } (commutative laws)
$E_4 : (P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$
$E_5 : (P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ } (associative laws)
$E_6 : P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
$E_7 : P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ } (Distributive laws)
$E_8 : \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
$E_9 : \neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ } (De Morgan's laws)
$E_{10} : P \vee P \Leftrightarrow P$
$E_{11} : P \wedge P \Leftrightarrow P$
$E_{12} : R \vee (P \wedge \neg P) \Leftrightarrow R, R \vee \mathbb{F} \Leftrightarrow R$
$E_{13} : R \wedge (P \vee \neg P) \Leftrightarrow R, R \wedge \mathbb{T} \Leftrightarrow R$
$E_{14} : R \vee (P \vee \neg P) \Leftrightarrow \mathbb{T}$
$E_{15} : R \wedge (P \wedge \neg P) \Leftrightarrow \mathbb{F}$
$E_{16} : P \rightarrow Q \Leftrightarrow \neg P \vee Q$

$E_{17}: \neg(P \to Q) \Leftrightarrow P \wedge \neg Q$

$E_{18}: P \to Q \Leftrightarrow \neg Q \to \neg P$

$E_{19}: P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$

$E_{20}: \neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$

$E_{21}: P \rightleftarrows Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

$E_{22}: (P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

Rules of Inference:

A set of premises $H_1, H_2, H_3, \ldots, H_m$ and a conclusion C are given

We assume that $H_1, H_2, H_3 \ldots, H_m$ are all true and we want to conclude the conclusion. That is we want to conclude that the conclusion C follows logically from the premises $H_1, H_2, H_3 \ldots H_m$.

1. Rule P:

A premise may be introduced at any point in the derivation.

2. Rule T:

A formula S may be introduced in a derivation if S is tautology implied by any one (or) more of the Preceding formulas in the derivation.

Validity using truth table technique:

1. Determine whether the conclusion C follows logically from the premises H_1 and H_2.

a) $H_1: P \to Q$	$H_2: P$	$C: Q$
b) $H_1: P \to Q$	$H_2: \neg P$	$C: Q$
c) $H_1: P \to Q$	$H_2: \neg(P \wedge Q)$	$C: \neg P$
d) $H_1: \neg P$	$H_2: P \rightleftarrows Q$	$C: \neg(P \wedge Q)$
e) $H_1: P \to Q$	$H_2: Q$	$C: P$

Soln:

P	Q	$\neg P$	$P \to Q$	$P \rightleftarrows Q$	$\neg(P \wedge Q)$
T	T	F	T	T	F
T	F	F	F	F	T
F	T	T	T	F	T
F	F	T	T	T	T

a) $H_1: P \to Q$ $H_2: P$ $C: Q$

The conclusion Q follows logically from the given Premises H_1 and H_2.

b) $H_1: P \to Q$ $H_2: \neg P$ $C: Q$

Here in the third row, both the premises are true and the conclusion is also true and in the fourth row both the premises are true but the conclusion is false.

$\therefore$ The conclusion Q does not follow logically from the given premises $P \to Q$ and $\neg P$.

c) $H_1: P \to Q$ $H_2: \neg(P \wedge Q)$ $C: \neg P$

The conclusion $\neg P$ follows logically from the given Premises

$P \to Q$ and $\neg(P \wedge Q)$

D) $H_1: \neg P$ $H_2: P \rightleftarrows Q$ $C: \neg(P \wedge Q)$

The conclusion $\neg(P \wedge Q)$ follows logically from the given Premises H_1 and H_2.

e) $H_1: P \to Q$ $H_2: Q$ $C: P$

The conclusion P does not follows the given Premises

$P \to Q$ and Q.

Q. Demonstrate that R is a valid Inference from the Premises $P \to Q$, $Q \to R$ and P.

Soln:

Premises : $P \to Q$, $Q \to R$ and P

conclusion : R

$\{1\}$	(1) P	Rule P
$\{2\}$	(2) $P \to Q$	Rule P
$\{1,2\}$	(3) Q	Rule T, (1), (2), and I_{11}

$\{4\}$ (4) $Q \to R$ Rule P

$\{1,2,4\}$ (5) R Rule T (3), (4) and I_{11}

where $I_{11}: P, P \to Q \Rightarrow Q$

$\therefore R$ is a valid iforence from the given premises $P \to Q$, $Q \to R$ and P.

3. S.T $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to R \vee S$

soln:

Premises: $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to R \vee S$

conclusion : $R \vee S$

$\{1\}$	(1)	$(C \vee D) \to \neg H$	Rule P
$\{2\}$	(2)	$\neg H \to (A \wedge \neg B)$	Rule P
$\{1,2\}$	(3)	$(C \vee D) \to (A \wedge \neg B)$	Rule T (1), (2) and I_{13}
$\{4\}$	(4)	$(A \wedge \neg B) \to R \vee S$	Rule P
$\{1,2,4\}$	(5)	$(C \vee D) \to R \vee S$	Rule T (3), (4) and I_{13}
$\{6\}$	(6)	$C \vee D$	Rule P
$\{1,2,4,6\}$	(7)	$R \vee S$	Rule T (5), (6) and I_{11}

where $I_{11}: P, P \to Q \Rightarrow Q$

$I_{13}: P \to Q, Q \to R \Rightarrow P \to R$

$\therefore R \vee S$ follows logically from the given premises.

4. S.T $S \vee R$ is Tautology implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

soln:

Premises: $P \vee Q$, $P \to R$, $Q \to S$

conclusion : $S \vee R$

$\{1\}$	(1)	$P \vee R$	Rule P
$\{1\}$	(2)	$\neg P \to Q$	Rule T (1) and E_{16}
$\{3\}$	(3)	$Q \to S$	Rule P
$\{1,3\}$	(4)	$\neg P \to S$	Rule T (2), (3) and I_{13}
$\{1,3\}$	(5)	$\neg S \to P$	Rule T (4) and E_{18} and E_1

$\{6\}$ (6) $P \to R$ Rule P

$\{1,3,6\}$ (7) $\neg S \to R$ Rule T (5), (6) and I_{13}

$\{1,3,6\}$ (8) $S \vee R$ Rule T (7), E_{16} and E_1

where $I_{13}: P \to Q, Q \to R \Rightarrow P \to R$

$E_1: \neg\neg P \Leftrightarrow P$

$E_{16}: P \to Q \Leftrightarrow \neg P \vee Q$

$E_{18}: P \to Q \Leftrightarrow \neg Q \to \neg P$

$\therefore S \vee R$ is tautology implied by

$(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

5. S.T $I_{12}: \neg Q, P \to Q \Rightarrow \neg P$

soln:

Premises: $\neg Q, P \to Q$

conclusion: $\neg P$

$\{1\}$ (1) $P \to Q$ Rule P

$\{1\}$ (2) $\neg Q \to \neg P$ Rule T (1) and E_{18}

$\{3\}$ (3) $\neg Q$ Rule P

$\{1,3\}$ (4) $\neg P$ Rule T (2), (3) and I_{11}

where

$I_{11}: P, P \to Q \Rightarrow Q$

$E_{18}: P \to Q \Leftrightarrow \neg Q \to \neg P$

6. S.T $R \wedge (P \vee Q)$ is a valid inference from the premises $P \vee Q, Q \to R, P \to M$ and $\neg M$.

soln:

Premises: $P \vee Q, Q \to R, P \to M$ and $\neg M$.

conclusion: $R \wedge (P \vee Q)$

$\{1\}$ (1) $P \to M$ Rule P

$\{2\}$ (2) $\neg M$ Rule P

$\{1,2\}$ (3) $\neg P$ Rule T (1), (2) and I_{12}

$\{4\}$ (4) $P \vee Q$ Rule P

$\{1,2,4\}$ (5) Q Rule T (3), (4) and I_{10}

{6} (6) $Q \to R$ Rule P

{1,2,4,6} (7) R Rule T (5),(6) and I_{11}

{1,2,4,6} (8) $R \wedge (P \vee Q)$ Rule T (4),(7) and I_9

where, $I_{12}: \neg Q, P \to Q \Rightarrow \neg P$

$I_{10}: \neg P, P \vee Q \Rightarrow Q$

$I_{11}: P, P \to Q \Rightarrow Q$

$I_9: P, Q \Rightarrow P \wedge Q$

Hence $R \wedge (P \vee Q)$ is a valid inference from the Premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

7. If there was a ball game then travelling was difficult. If they arrived on time then travelling was not difficult. They arrived on time. Therefore there was no ball game. S.T these statements constitude a valid argument.

soln: Let us define

P: There was a ball game

Q: Travelling was difficult

R: They arrived on time

Premises : $P \to Q$, $R \to \neg Q$, R

conclusion : $\neg P$

{1} (1) $R \to \neg Q$ Rule P

{2} (2) $P \to Q$ Rule P

{1,2} (3) $\neg Q \to \neg P$ Rule T (2) and E_{18}

{1,2} (4) $R \to \neg P$ Rule T (1), (3) and I_{13}

{5} (5) R Rule P

{1,2,5} (6) $\neg P$ Rule T (4),(5) and I_{11}

where, $E_{18}: P \to Q \Leftrightarrow \neg Q \to \neg P$

$I_{13}: P \to Q, Q \to R \Rightarrow P \to R$

$I_{11}: P, P \to Q \Rightarrow Q$

$\therefore$ The given statements constitute a valid argument.

Rule CP (or) conditional Proof:

If we derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.

Rule CP is also called the deduction theorem and is generally used if the conclusion is of the form $R \to S$.

In such cases, R is taken as an additional Premise and S is derived from the given premises and R.

1. S.T $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ and Q.

Soln:

Premises : $P \to (Q \to S)$, $\neg R \vee P$ and Q

conclusion : $R \to S$

Since the conclusion is $R \to S$, we use rule CP

Therefore we include R as an additional premise and we show S first.

$\therefore$ Premises : $P \to (Q \to S)$, $\neg R \vee P$, Q, and R (additional premise)

conclusion : S

$\{1\}$	(1)	$\neg R \vee P$	Rule P
$\{2\}$	(2)	R	Rule P (additional premise)
$\{1,2\}$	(3)	P	Rule T (1),(2) and I_{10}, E_1
$\{4\}$	(4)	$P \to (Q \to S)$	Rule P
$\{1,2,4\}$	(5)	$Q \to S$	Rule T (3),(4) and I_{11}
$\{6\}$	(6)	Q	Rule P
$\{1,2,4,6\}$	(7)	S	Rule T (5),(6) and I_{11}
$\{1,4,6\}$	(8)	$R \to S$	Rule CP

2. S.T the following statements constitute a valid argument

i) If A works hard then either B or C will enjoy themselves.

ii) If B enjoys himself then A will not work hard

iii) If D enjoys himself then C will not enjoys himself

Therefore if A works hard then D will not enjoy himself.

Soln: Let us define

A : A works hard

B : B enjoys himself

C : C enjoys himself

D : D enjoys himself

Premices: $A \to (B \vee C)$, $B \to \neg A$, $D \to \neg C$

conclusion: $A \to \neg D$

Since the conclusion is $A \to \neg D$, we use rule CP

Therfore we include A as an additional premise and we show $\neg D$ first.

Premises: $A \to (B \vee C)$, $B \to \neg A$, $D \to \neg C$, A

conclusion: $\neg D$

{1}	(1)	$A \to (B \vee C)$	Rule P
{2}	(2)	A	Rule P (additional Premise)
{1,2}	(3)	$B \vee C$	Rule T (1),(2) and I_{11}
{4}	(4)	$B \to \neg A$	Rule P
{4}	(5)	$A \to \neg B$	Rule T (4) and E_{18}, E_1
{2,4}	(6)	$\neg B$	Rule T (2),(5) and I_4
{1,2,4}	(7)	C	Rule T (3),(6) and I_{10}
{8}	(8)	$D \to \neg C$	Rule P
{1,2,4,8}	(9)	$\neg D$	Rule T (7),(8) and I_{12}
{1,4,8}	(10)	$\neg D$	Rule CP

where, $I_{10}: \neg P, P \vee Q \Rightarrow Q$

$I_{11}: P, P \to Q \Rightarrow Q$

$I_{12}: \neg Q, P \to Q \Rightarrow \neg P$

$E_{18}: P \to Q \Leftrightarrow \neg Q \to \neg P$

$E_1: \neg\neg P \Leftrightarrow P$