PARTIAL DIFFERENTIAL EQUATIONS

UNIT-I Second order partial Differential Equations

origin of second order partial differential equations - Linear differential equations with constant coefficients - Method of solving partial (linear) differential equation - Classification of second order partial differential equations - Canonical forms - Adjoint operators - Riemann method. (chapter 2: Sections 2.1 to 2.5)

UNIT-II Elliptic Differential Equations:

Elliptic differential equations - occurrence of Laplace and poisson equations - Boundary value Problems - Separation of variables method - Laplace equation in cylindrical - spherical co-ordinates, Dirichlet and Neumann problems for circle - sphere (chapter 3: Sections 3.1 to 3.9)

UNIT-III parabolic Differential Equations:

parabolic differential equations - occurrence of the diffusion equation - Boundary condition - Separation of variable method - Diffusion equation in cylindrical - spherical co-ordinates (chapter 4: sections 4.1 to 4.5)

UNIT-IV Hyperbolic Differential Equations:

Hyperbolic differential equations - occurrence of wave equation - one dimensional wave equations - Reduction to canonical form - D'Alembertz solution - separation of variable method - periodic solutions - cylindrical - spherical co-ordinates - Duhamel principle for wave equations. (chapter 5: sections 5.1 to 5.6 and 5.9)

UNIT-V Integral Transform:

Laplace transforms - solution of partial differential equation - Diffusion equation - wave equation - Fourier transform - Application to partial differential equation - Diffusion equation - wave equation - Laplace equation (chapter 6: sections 6.2 to 6.4).

TEXT BOOK

1. J.N. Sharma and K. singh, partial Differential Equation for Engineers and scientists, Narosa publ. House, chennai, 2001.

BOOKS FOR REFERENCE

1. I.N. Snedden, Elements of partial Differential Equations, McGraw Hill, New york 1964.
2. K. Sankar Rao, Introduction to partial Differential Equations, prentice Hall of India, New Delhi, 1995.
3. S.J. Farlow, partial Differential Equations for scientists and Engineers, John wiley Sons, New York, 1982.

UNIT-I

Introduction:

Many problems in physics and engineering while formulated Mathematically give rise to PDE. To understand the physical behaviour of Mathematical Model on should have some knowledge of Mathematical charectors, Properties and the knowledge of finding solutions to the PDE. The knowledge of finding solutions of PDE is absolutely necessary in order to have the depth understanding of the subjects like Fluid dynamics, Aero dynamics, Heat transfer, Electricity, Thermo electricity, sesimology, waves and Electro magnets.....

Second order partial differential Equations:

origin of second order PDE:

Let $z = f(u) + g(v) + w$ — (1) be a function where f and g are arbitrary functions of u and v respectively. and u, v and w are functions of x and y.

Diff. ① P.W.r to x,

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q$$

$$p = f'(u)\, u_x + g'(v)\, v_x + w_x \quad \text{— ②}$$

$$\frac{\partial^2 z}{\partial x \partial x} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s$$

Diff. ① P.W.r. to y,

$$q = f'(u)\, u_y + g'(v)\, v_y + w_y \quad \text{— ③} \qquad \frac{\partial^2 z}{\partial y \partial y} = t.$$

Diff. (2) P.W.r to x,

$$r = f'(u)\, u_{xx} + f''(u)\, u_x^2 + g'(v)\, v_{xx} + g''(v)\, v_x^2 + w_{xx} \quad \text{— ④}$$

Diff. (2) P.W.r to y.

$$s = f'(u)\, u_{xy} + f''(u)\, u_x u_y + g'(v)\, v_{xy} + g''(v)\, v_x v_y + w_{xy} \quad \text{— ⑤}$$

Diff. (3) P.W.r to y

$$t = f'(u)\, u_{yy} + f''(u)\, u_y^2 + g'(v)\, v_{yy} + g''(v)\, v_y^2 + w_{yy} \quad \text{— ⑥}$$

Eqns ② to ⑥ contain arbitrary quantities f', g', f'', g'', eliminating these quantities, we have

$$\begin{vmatrix} p - w_x & u_x & v_x & 0 & 0 \\ q - w_y & u_y & v_y & 0 & 0 \\ r - w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s - w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t - w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0.$$

which involve p, q, r, s, t and known functions of x and y.

Expanding interms of first column, we have

$$Rr + Ss + Tt + Pp + Qq = W \quad \text{—— (7)}$$

where R, S, T, P, Q, W are known functions of x and y.

$\therefore$ Eqn (1) is the solution of second order linear partial differential equation (7) which is a particular type of eqn and contains dependent variable z.

Problem:

(1) If $u = f(x+iy) + g(x-iy)$, where f and g are arbitrary functions, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution:

$$u = f(x+iy) + g(x-iy) \quad \text{—— (1)}$$

Diff (1) p.w.r. to x

$$\frac{\partial u}{\partial x} = f'(x+iy) + g'(x-iy) \quad \text{—— (2)}$$

Diff. (2) p.w.r to x

$$\frac{\partial^2 u}{\partial x^2} = f''(x+iy) + g''(x-iy) \quad \text{—— (3)}$$

Diff. (1) p.w.r to y

$$\frac{\partial u}{\partial y} = if'(x+iy) - ig'(x-iy) \quad \text{—— (4)}$$

Diff (4) p.w.r. to y

$$\frac{\partial^2 u}{\partial y^2} = -f''(x+iy) - g''(x-iy) \quad \text{—— (5)}$$

Adding (3) and (5)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2 If $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ and f, g are arbitrary functions of x, t, y, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, provided $\alpha^2 = 1 - \frac{v^2}{c^2}$.

Solution :

$$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y) \quad \text{--- (1)}$$

Diff. (1) P.W.r to x

$$\frac{\partial u}{\partial x} = f'(x - vt + i\alpha y) + g'(x - vt - i\alpha y) \quad \text{--- (2)}$$

Diff. (2) P.W.r to x

$$\frac{\partial^2 u}{\partial x^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \quad \text{--- (3)}$$

Diff. (1) P.W.r to y

$$\frac{\partial u}{\partial y} = f'(x - vt + i\alpha y)(i\alpha) + g'(x - vt - i\alpha y)(-i\alpha)$$

$$= i\alpha f'(x - vt + i\alpha y) - i\alpha g'(x - vt - i\alpha y) \quad \text{--- (4)}$$

Diff. (4) P.W.r to y

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2 f''(x - vt + i\alpha y) - \alpha^2 g''(x - vt - i\alpha y) \quad \text{--- (5)}$$

Diff. (1) P.W.r to t

$$\frac{\partial u}{\partial t} = f'(x - vt + i\alpha y)(-v) + g'(x - vt - i\alpha y)(-v)$$

$$= -Vf'(x-Vt+i\alpha y) - Vg'(x-Vt-i\alpha y) \quad \text{--- (6)}$$

Diff. (6) p.w.r to t

$$\frac{\partial^2 u}{\partial t^2} = V^2 f''(x-Vt+i\alpha y) + V^2 g''(x-Vt-i\alpha y) \quad \text{--- (7)}$$

Adding (3) and (5)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(x-Vt+i\alpha y)(1-\alpha^2) + g''(x-Vt-i\alpha y)(1-\alpha^2)$$

$$= (1-\alpha^2)\left[f''(x-Vt+i\alpha y) + g''(x-Vt-i\alpha y)\right]$$

From (7) $\quad = (1-\alpha^2)\dfrac{1}{V^2}\dfrac{\partial^2 u}{\partial t^2}$

$$= \frac{1-\alpha^2}{V^2}\frac{\partial^2 u}{\partial t^2}$$

Since $\alpha^2 = 1 - \dfrac{V^2}{c^2} \Rightarrow \alpha^2 - 1 = -\dfrac{V^2}{c^2}$

$$\frac{\alpha^2-1}{-V^2} = \frac{1}{c^2}$$

$$\frac{1-\alpha^2}{V^2} = \frac{1}{c^2}$$

$$\Rightarrow \quad = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Linear partial Differential Equations with Constant co-efficients:

An equation of the form

$$F(D, D')z = f(x,y) \quad \text{——} (1)$$

where $F(D, D') = \sum_r \sum_s C_{rs} D^r D'^s$ is a differential operator, where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$ and C_{rs} are constants is known as linear Partial differential equation with constant coefficients.

Solution of Linear PDE with constant coefficients

We know that $F(D, D')z = f(x,y)$
where $F(D, D') = \sum_r \sum_s C_{rs} D^r D'^s$ is a Linear PDE with constant coefficient.

Any most general solution of the homogeneous linear PDE.

$F(D, D')z = 0$ is known as the complementry function of (1). The particular solution of (1) which is known as particular integral of (1) which contains no arbitrary constants and arbitrary functions.

The general solution of (1) is the sum of C.F and P.I.

Theorem:

If $u_1, u_2, \ldots, u_n$ are solutions of homogeneous linear partial differential equation

$$F(D, D')z = 0$$

Then $\sum_{r=1}^{n} c_r u_r$, where c_r's are arbitrary constants is also a solution.

Proof.

Given u_r, $r = 1, 2, \ldots, n$ are solution of

$$F(D, D')z = 0$$

$$F(D, D')u_r = 0$$

$$\therefore F(D, D')c_r u_r = c_r F(D, D')u_r$$

Now, $F(D, D')\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} F(D, D')u_r$

$$\Rightarrow F(D, D')\sum_{r=1}^{n} c_r u_r = \sum_{r=1}^{n} F(D, D')(c_r u_r)$$

$$= \sum_{r=1}^{n} c_r F(D, D')u_r$$

$$= 0 \qquad (\because F(D, D')u_r = 0).$$

$\therefore \sum_{r=1}^{n} c_r u_r$ is the solution of $F(D, D')z = 0$.

Defn:

Reducible and irreducible operators

The operator $F(D, D')$ is said to be a reducible operator if it can be factorized into the linear factors of the type. $D+aD'+b$ where a and b are constants.

It is irreducible if it is not reducible

For example,

$D^2 - D'^2$ is a reducible operator.

since $D^2 - D'^2 = (D+D')(D-D')$

$D^2 + D'^2$ is an irreducible operator.

A PDE $F(D, D')z = f(x,y)$ is reducible, if $F(D, D')$ is reducible otherwise it is irreducible.

Theorem

If $\alpha_r D + \beta_r D' + \gamma_r$ is a factor of $F(D, D')$ and ϕ_r is an arbitrary function of the single variable. Then $u_r = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y)$ for $\alpha_r \neq 0$ is a solution of the equation $F(D, D')z = 0$.

Proof:

To prove $u_r = e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)} \phi_r(\beta_r x - \alpha_r y), \ \alpha_r \neq 0$ — (1)

the solution of $F(D,D')z=0$

Diff (1) p.w.r to x

$$Du_r = \frac{-\gamma_r}{\alpha_r} e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)} \phi_r(\beta_r x - \alpha_r y) + e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)} \phi_r'(\beta_r x - \alpha_r y)\beta_r$$

$$\Rightarrow Du_r = \frac{-\gamma_r}{\alpha_r} u_r + \beta_r e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)} \phi_r'(\beta_r x - \alpha_r y)$$

Diff. (1) p.w.r. to y.

$$D'u_r = -\alpha_r e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)} \phi_r'(\beta_r x - \alpha_r y)$$

Now,

$$(\alpha_r D + \beta_r D' + \gamma_r) u_r = -\gamma_r u_r + \alpha_r \beta_r e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)} \phi_r'(\beta_r x - \alpha_r y)$$
$$- \alpha_r \beta_r e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)} \phi_r'(\beta_r x - \alpha_r y) + \gamma_r u_r$$

$$(\alpha_r D + \beta_r D' + \gamma_r) u_r = 0$$

$$\text{Now, } F(D,D')u_r = \prod_{\substack{s=1 \\ r\neq s}}^{n} (\alpha_r D + \beta_r D' + \gamma_r)(\alpha_r D + \beta_r D' + \gamma_r) u_r$$

$$= 0$$

i.e., $u_r = \exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \phi_r(\beta_r x - \alpha_r y)$ is the solution of $F(D,D')z=0$.

Hence the proof.

Defn:

A PDE $F(D,D')z = f(x,y)$ is said to be reducible if $F(D,D')z = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r)$

Method of Solving Linear PDE

Solution of reducible equation.

Let $F(D, D')z = f(x,y)$ be a reducible PDE

Since it is reducible,

$$F(D, D')z = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r) z$$

In order to find the complementry function

we have, $(\alpha_r D + \beta_r D' + \gamma_r) z = 0$

$$\Rightarrow \alpha_r \frac{\partial z}{\partial x} + \beta_r \frac{\partial z}{\partial y} + \gamma_r z = 0$$

$$\Rightarrow \alpha_r p + \beta_r q = -\gamma_r z$$

This is of the form $Pp + Qq = R$

$\therefore$ soln is given by

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z} \qquad \text{— (I)}$$

considering $\frac{dx}{\alpha_r} = \frac{dy}{\beta_r}$

$$\beta_r dx = \alpha_r dy$$

$$\int \beta_r dx = \int \alpha_r dy + c_r$$

$$\beta_r x = \alpha_r y + c_r$$

$$\Rightarrow \beta_r x - \alpha_r y = c_r$$

considering, $\frac{dx}{\alpha_r} = \frac{dz}{-\gamma_r z}$

$$-\frac{\gamma_r}{\alpha_r} dx = \frac{dz}{z}$$

$$\int -\frac{\gamma_r}{\alpha_r} dx = \int \frac{dz}{z}$$

$$-\frac{\gamma_r}{\alpha_r} x = \log z + \log A_r$$

$$\Rightarrow \quad z = A_r e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)}$$

$$\therefore z = \phi_r(c_r) \exp\left(-\frac{\gamma_r x}{\alpha_r}\right)$$

$$z = \phi_r(\beta_r x - \alpha_r y) \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)$$

Suppose $\alpha_r \neq 0$, the complementry function.

$$C.F = \sum_{r=1}^{n} \phi_r(\beta_r x - \alpha_r y) \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)$$

where ϕ_r is an arbitrary function.

Particular case: $\alpha_r = 0$

(I) becomes $\frac{dx}{0} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z}$
(i) (ii) (iii)

From (i) & (ii)

$$\frac{dx}{0} = \frac{dy}{\beta_r}$$

$$\beta_r dx = 0$$

$$\int \beta_r dx = 0$$

$$\beta_r x = c_r$$

$$\frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z}$$

$$\Rightarrow -\gamma_r\, dy = \beta_r \frac{dz}{z}$$

$$\int -\gamma_r\, dy = \int \beta_r \frac{dz}{z}$$

$$-\gamma_r y = \beta_r \log z$$

$$\log z = \frac{-\gamma_r}{\beta_r} y$$

$$z = e^{\frac{-\gamma_r y}{\beta_r}}$$

$$z = \phi_r(\beta_r x) \exp\left(\frac{-\gamma_r y}{\beta_r}\right)$$

$$\therefore CF = \sum_{r=1}^{n} \phi_r(\beta_r x) \exp\left(\frac{-\gamma_r y}{\beta_r}\right)$$

The above two cases are applicable when there is no repeated factor of the type $\alpha_r D + \beta_r D' + \gamma_r$.

Case of repeated factors.

Suppose the PDE $F(D, D')z = 0$ has repeated factors.

Suppose that $(\alpha_r D + \beta_r D' + \gamma_r)^2$ is a factor of $F(D, D')$

Thus, we have

$$(\alpha_r D + \beta_r D' + \gamma_r)^2 z = 0 \quad \text{--- (1)}$$

Assume $(\alpha_r D + \beta_r D' + \gamma_r) z = z_1$ --- (2)

$$① \Rightarrow (\alpha_r D + \beta_r D' + \gamma_r) z_1 = 0 \quad \text{---} ③$$

By previous case,

$$z_1 = \phi_r(\beta_r x - \alpha_r y)\, e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)}$$

$$② \Rightarrow (\alpha_r D + \beta_r D' + \gamma_r) z = \phi_r(\beta_r x - \alpha_r y)\, e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)}$$

$$\Rightarrow (\alpha_r D + \beta_r D') z = \phi_r(\beta_r x - \alpha_r y)\, e^{-\left(\frac{\gamma_r z}{\alpha_r}\right)} - \gamma_r z \quad \text{---} ④$$

$\therefore$ Auxiliary eqn of ④ will be

$$\frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{\phi_r(\beta_r x - \alpha_r y)\, e^{-\left(\frac{\gamma_r z}{\alpha_r}\right)} - \gamma_r z}$$

From $\frac{dx}{\alpha_r} = \frac{dy}{\beta_r}$

$$\int \beta_r dx = \int \alpha_r dy$$

$$\beta_r x = \alpha_r y + c_r$$

$$\beta_r x - \alpha_r y = c_r$$

From

$$\frac{dx}{\alpha_r} = \frac{dz}{\phi_r(\beta_r x - \alpha_r y)\, e^{-\left(\frac{\gamma_r z}{\alpha_r}\right)} - \gamma_r z}$$

$$\Rightarrow \frac{dz}{dx} + \frac{\gamma_r z}{\alpha_r} = \frac{1}{\alpha_r} \phi_r(c_r)\, e^{\left(-\frac{\gamma_r x}{\alpha_r}\right)} \quad \text{---} ⑤$$

$$I.F = e^{\int p\,dx} = e^{\int \frac{\gamma_r}{\alpha_r} dx} = e^{\frac{\gamma_r x}{\alpha_r}}$$

Solution of (5)

$$z\, e^{\left(\frac{\gamma_r x}{\alpha_r}\right)} = \int \frac{1}{\alpha_r} \phi_r(c_r)\, e^{-\left(\frac{\gamma_r x}{\alpha_r}\right)} e^{\left(\frac{\gamma_r x}{\alpha_r}\right)} dx$$

$$= \int \frac{1}{\alpha_r} \phi_r(c_r)\, dx$$

$$= \frac{1}{\alpha_r} \phi_r(c_r)\, x + \underbrace{\phi_r(c_r)}_{\text{Integral constant}}$$

$$\therefore \quad z = e^{-\frac{\gamma_r x}{\alpha_r}} \left[\phi_r(\beta_r x - \alpha_r y) + \phi_r(\beta_r x - \alpha_r y) \right]$$

This procedure can be generalized upto any order of repetition of factors adding this to the sum of the other solutions corresponding to the linear factors without repetition, we get the required complementary function.

Solution of Irreducible equations with constant coefficients.

Let $F(D, D') = f(x,y)$ be an irreducible linear PDE with constant coefficients.

Let $F(D, D') = F_1(D, D')\, F_2(D, D')$

where F_2 is reducible and F_1 is irreducible.

Then the solution of corresponding linear factors of $F_2(D,D')$ can be obtained as

$$e^{\frac{-\gamma_r x}{\alpha_r}}\,\phi_r(\beta_r x-\alpha_r y) \quad \text{if } \alpha_r \neq 0.$$

$$e^{\frac{-\gamma_r x}{\alpha_r}}\,\phi_r(\beta_r x) \quad \text{if } \alpha_r = 0.$$

In order to find the solutions corresponding to the irreducible factors of $F_1(D,D')$ we assume that $z=e^{ax+by}$ as solution.

i.e., $F_1(D,D')z=0$

$\Rightarrow F_1(D,D')\,e^{ax+by}=0$

$\Rightarrow F_1(a,b)\,e^{ax+by}=0$

$\Rightarrow F_1(a,b)=0$

$\therefore z=\sum_r c_r\,e^{a_r x+b_r y}$

where $F_1(a_r,b_r)=0$, $r=1,2,3,\ldots$ is a complementary function corresponding to the irreducible factors.

<u>Rules for finding complementary functions.</u>

The procedure for finding C.F can be extended to differential equations of higher orders.

consider the equation

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$$

$$D^2 + a_1 D D' + a_2 D'^2 = 0 \quad \text{--- (1)}$$

Auxiliary equation is given by

$$m^2 + a_1 m + a_2 = 0 \quad , \quad m = \frac{D}{D'}$$

case (i)

Let the roots of A.E be m_1 and m_2

$m_1 \neq m_2$

$$(D - m_1 D')(D - m_2 D')z = 0$$

Now, consider $(D - m_2 D')z = 0$

$$\Rightarrow p - m_2 q = 0$$

A.E $\quad \frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}$

considering, $\frac{dx}{1} = \frac{dy}{-m_2}$

$$\Rightarrow -m_2 dx = dy$$

$$dy + m_2 dx = 0$$

$\int$ing

$$\int dy + m_2 \int dx = 0$$

$$y + m_2 x = c_2$$

considering, $\frac{dx}{1} = \frac{dz}{0}$

$$dz = 0$$

$$\Rightarrow z = c_1$$

$\therefore$ solution of $(D-m_2D')z=0$ is given by

$$z=f_2(y+m_2x)$$

considering $(D-m_1D')z=0$, we get the solution

$$z=f_1(y+m_1x)$$

$\therefore$ complete solution of $\frac{\partial^2 z}{\partial x^2}+a_1\frac{\partial^2 z}{\partial x\partial y}+a_2\frac{\partial^2 z}{\partial y^2}=0$ is given by

$$z=f_1(y+m_1x)+f_2(y+m_2x)$$

Where f_1, f_2 are arbitrary functions.

<u>case(ii)</u> $m_1=m_2=m$ $\rightarrow$ $(D-m_1D')(D-m_2D')z=0$

$(D-mD')(D-mD')z=0$

$(D-mD')^2z=0$ (from ①) $(D-mD')^2z=0$

Let $(D-mD')z=u$ —— ⊛

$\therefore (D-mD')u=0$ —— (I)

of I, $\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{0}$

considering, $\frac{dx}{1}=\frac{dy}{-m}$

$$-mdx=dy$$

$$dy+mdx=0$$

Jing $\qquad y+mx=c_1$

$\therefore$ solution of (I) is $u=f(y+mx)$ using the value of u in ⊛.

$$(D-mD')z=f(y+mx)$$

$$\Rightarrow p-mq=f(y+mx)$$

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(c_1)} \quad (\because c_1 = y+mx)$$

$$\frac{dx}{1} = \frac{dz}{f(c_1)}$$

$$f(c_1)\,dx = dz$$

Sing $z = xf(c_1) + c_2$

$$z = xf(y+mx) + c_2$$

$\therefore$ complete solution is given by

$$z = f_1(y+mx) + xf_2(y+mx)$$

<u>Note:</u>

(i) If the roots of A.E are $m_1, m_2, m_3, \ldots$ all distinct, then the complementary function

$= f_1(y+m_1x) + f_2(y+m_2x) + \cdots$

where $f_1, f_2, \ldots$ are arbitrary functions

(ii) If two roots of A.E are equal

i.e., $m_1 = m_2$, then the C.F is

$= f_1(y+m_1x) + xf_2(y+m_1x) + f_3(y+m_3x) + \cdots$

(iii) If three roots of A.E are equal,

i.e., $m_1 = m_2 = m_3$, then the C.F is

$= f_1(y+m_1x) + xf_2(y+m_1x) + x^2 f_3(y+m_1x) + f_4(y+m_4x)$

$+ \cdots$

Rules for finding particular integral.

The Particular Integral of the equation

$\phi(D, D')z = F(x, y)$

where $\phi(D, D') = D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots$

$+ a_n D'^n$ is given by $\frac{1}{\phi(D, D')} F(x, y)$.

case(i)

Short methods to find P.I

— $F(x, y) = e^{ax+by}$

then $P.I = \frac{1}{\phi(D, D')} e^{ax+by}$

Rule: Replace D by a and D' by b.

$P.I = \frac{1}{\phi(a, b)} e^{ax+by}$

Provided $\phi(a, b) \neq 0$.

If $\phi(a, b) = 0$ —

case (ii)

$F(x, y) = \sin(ax+by)$ (or) $\cos(ax+by)$

$$P.I = \frac{1}{\phi(D, D')} \sin(ax+by)$$

$$= \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax+by)$$

provided $\phi(-a^2, -ab, -b^2) \neq 0$.
otherwise, it is called the case of failure.
A similar rule holds for $\cos(ax+by)$

Case (iii)

$$F(x,y) = x^m y^n$$

$$P.I = \frac{1}{\phi(D,D')} x^m y^n$$, where m,n are positive integers, then

$$P.I = [\phi(D,D')]^{-1} x^m y^n$$

If $m<n$, we expand binomially $[\phi(D,D')]^{-1}$ in powers of $\frac{D}{D'}$ and for $m>n$, we expand binomially $[\phi(D,D')]^{-1}$ in powers of $\frac{D'}{D}$

Also, we have $\frac{1}{D} F(x,y) = \int\limits_{y\ \text{constant}} F(x,y)\,dx$ and

$$\frac{1}{D'} F(x,y) = \int\limits_{x\ \text{constant}} F(x,y)\,dy.$$

General Method to find Particular Integral

This method is applicable to all the cases where $F(x,y)$ is not of the form discussed above and the short cut method fails.

Suppose $\phi(D, D')$ can be factored as n-linear factors such that

$$\phi(D, D') = (D - m_1 D')(D - m_2 D') \dots (D - m_n D')$$

The $P.I = \frac{1}{\phi(D, D')} F(x, y)$

$$= \frac{1}{(D - m_1 D') \dots (D - m_n D')} F(x, y)$$

consider $\frac{1}{(D - mD')} F(x, y)$

Now, consider the equation

$$(D - mD')z = F(x, y)$$

$$\Rightarrow p - mq = F(x, y)$$

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{F(x, y)}$$

Now, $\frac{dx}{1} = \frac{dy}{-m}$

$$\Rightarrow -m\,dx = dy$$

$\int$ing $\quad y + mx = c$

From $\frac{dx}{1} = \frac{dz}{F(x, y)}$

$$F(x, y)\,dx = dz$$

$$F(x, c - mx)\,dx = dz$$

$\int$ing $\quad z = \int F(x, c - mx)\,dx$

i.e. $\frac{1}{(D - mD')} F(x, y) = \int F(x, c - mx)\,dx$

where c is replaced by $y+mx$ after integration.

Thus P.I can be evaluated by the repeated application of above rule.

Problem

Solve $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} - \dfrac{\partial^3 z}{\partial x \partial y^2} + 2\dfrac{\partial^3 z}{\partial y^3} = e^{x+y}$ —— (1)

Solution:

$$(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$$

$$\Rightarrow D^2(D-2D') - D'^2(D-2D')z = e^{x+y}$$

$$\Rightarrow (D^2 - D'^2)(D-2D')z = e^{x+y}$$

$$\Rightarrow (D-D')(D+D')(D-2D')z = e^{x+y} \quad \text{—— (2)}$$

To find C.F

consider $(D-D')z = 0$

$$p - q = 0$$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{0}$$

$$\frac{dx}{1} = \frac{dy}{-1} \Rightarrow dx + dy = 0$$

Integrating $x + y = c$

$\phi_1(x+y)$ is the solution of (2).

$(D-D')(D+D')(D-2D')=0$ —— (3)

$(D+D')z=0$

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{0}$$

$$\frac{dx}{1}=\frac{dy}{1}$$

$dx-dy=0$

Integrating, $x-y=c_2$

$\phi_2(x-y)$ is the solution of (3)

Now,

$(D-2D')z=0$

$p-2q=0$

$$\frac{dx}{1}=\frac{dy}{-2}=\frac{dz}{0}$$

$$\frac{dx}{1}=\frac{dy}{-2}$$

$-2dx=dy$

$\Rightarrow dy+2dx=0$

Integrating $y+2x=c$

$\therefore \phi_3(y+2x)$ is a solution of (3)

$\therefore C.F=\phi_1(y+x)+\phi_2(x-y)+\phi_3(y+2x)$

To find P.I

$$P.I = \frac{1}{\phi(D,D')} F(x,y)$$

$$= \frac{1}{(D-D')(D+D')(D-2D')} e^{x+y}$$

$$= \frac{1}{(D-D')(2)(-1)} e^{x+y}$$

$$= \frac{-1}{2(D-D')} e^{x+y}$$

Let $W = \frac{e^{x+y}}{D-D'}$

$$\therefore P.I = -\frac{W}{2}$$

$$\therefore P.I = -\frac{x e^{x+y}}{2}$$

consider $(D-D')W = e^{x+y}$

$$P-q = e^{x+y}$$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dW}{e^{x+y}}$$

$$\frac{dx}{1} = \frac{dy}{-1}$$

$$\Rightarrow dx+dy=0$$

Integrating, $x+y=c_1$

$$\frac{dx}{1} = \frac{dW}{e^{c_1}}$$

$$e^{c_1}dx = dW$$

Integrating, $xe^{c_1} = w$

$$\therefore \; P.I = \frac{-xe^{x+y}}{2}$$

$\therefore$ The solution of the given PDE is

$$z = CF + P.I.$$

$$z = \phi_1(x+y) + \phi_2(x-y) + \phi_3(y+2x) - \frac{xe^{x+y}}{2}.$$

— x —

<u>Alternate Method</u>

$$(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$$

A.E $\quad m^3 - 2m^2 - m + 2 = 0 \qquad D = m$

$D' = 1$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$m_1 = 1, \; m_2 = -1, \; m_3 = 2$

$$C.F = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+2x)$$

$x-y$

$$P.I = \frac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3} e^{x+y}$$

$$= \frac{x\, e^{x+y}}{3D^2 - 4DD' - D'^2}$$

$$= \frac{x e^{x+y}}{3-4-1}$$

$$P.I = \frac{x e^{x+y}}{-2}$$

$$\therefore z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+2x) - \frac{x e^{x+y}}{2}$$

— x —

(2) Solve $(D^2 - D')z = 2y - x^2$

Soln

$$(D^2 - D'^2)z = 2y - x^2$$

A.E $m^2 - 1 = 0$, $m^2 = 1$

$m = \pm 1$

$m_1 = 1$, $m_2 = -1$

$$C.F = \phi_1(y+x) + \phi_2(y-x)$$

$$P.I = \frac{1}{(D^2 - D')}(2y - x^2)$$

$$= \frac{1}{-D'\left(1 - \frac{D^2}{D'}\right)}(2y - x^2)$$

$$= \frac{-1}{D'}\left(1 - \frac{D^2}{D'}\right)^{-1}(2y - x^2)$$

$$= \frac{-1}{D'}\left[1 + \frac{D^2}{D'} + \frac{D^4}{D'^2} + \frac{D^6}{D'^3} + \cdots\right](2y - x^2)$$

$$= \frac{-1}{D'}\left[1+\frac{D^2}{D'}\right](2y-x^2)$$

$$= \frac{-1}{D'}\left[2y-x^2+\frac{1}{D'}(-2)\right]$$

$$= \frac{-1}{D'}\left[2y-x^2-2y\right]$$

$$= \frac{-1}{D'}(-x^2)$$

$$P.I = x^2y$$

$\therefore$ The complete solution is given by

$$z = C.F + P.I$$

$$z = \phi_1(y+x) + \phi_2(y-x) + x^2y.$$

— x —

(3) Find the solution of the equation $\nabla^2 z = e^{-x}\cos y$ which tends to 0 as $x \to \infty$ and has the value $\cos y$ when $x=0$.

Soln:

The given equation $\nabla^2 z = e^{-x}\cos y$ — (1)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-x}\cos y$$

$$(D^2 + D'^2)z = e^{-x}\cos y$$

Let $z = e^{ax+by}$ be the solution of $(D^2+D'^2)z = 0$

$$(D^2+D'^2)(e^{ax+by})=0$$

$$(a^2+b^2)\,e^{ax+by}=0$$

$$\therefore (D^2+D'^2)\,z=0$$

$$\Rightarrow a^2+b^2=0$$

$$\therefore C.F = \sum_{r=0}^{\infty} A_r e^{a_r x+b_r y}$$

Where $a_r^2+b_r^2=0$, A_r is a constant

$$P.I = \frac{1}{D^2+D'^2}\, e^{-x}\cos y$$

$$= \cos y \, \frac{e^{-x}}{D^2-1} \qquad (D'^2 \to -1)$$

$$= \cos y \, \frac{x e^{-x}}{2D}$$

$$P.I = \frac{-x}{2} e^{-x} \cos y$$

$\therefore$ complete solution is

$$z = C.F + P.I$$

$$= \sum_{r=0}^{\infty} A_r e^{a_r x+b_r y} - \frac{x}{2} e^{-x}\cos y$$

Given $z \to 0$ as $x \to \infty$

$\therefore$ a_r must be $-ve$

Let $a_r = -\lambda_r$, $\lambda_r > 0$

$a_r = -\lambda_r$

We have, $a_r^2 + b_r^2 = 0$

$$\lambda_r^2 + b_r^2 = 0$$

$$b_r^2 = -\lambda_r^2$$

$$b_r = \pm i\lambda_r$$

$$\therefore z = \sum_{r=0}^{\infty} A_r e^{-\lambda_r x \pm i\lambda_r y} - \frac{x}{2} e^{-x}\cos y.$$

Given when $x=0$, $z = \cos y$

$$z = \sum_{r=0}^{\infty} A_r e^{-\lambda_r x} e^{\pm i\lambda_r y} - \frac{x}{2} e^{-x}\cos y$$

$$z = \sum_{r=0}^{\infty} B_r e^{-\lambda_r x} \cos(\lambda_r y + \varepsilon_r) - \frac{x}{2} e^{-x}\cos y \quad -(*)$$

Given when $x=0$, $z = \cos y$

$$\therefore \cos y = \sum_{r=0}^{\infty} B_r \cos(\lambda_r y + \varepsilon_r)$$

$B_r = 1$, when $r=0$
$= 0$, when $r \neq 0$

$\lambda_r = 1$, if $r=0$
$= 0$. if $r \neq 0$

$\varepsilon_r = 0 \quad \forall r.$

Hence the soln is given by from $(*)$

$$z = e^{-x}\cos y - \frac{x}{2}e^{-x}\cos y$$

$z = \left(1-\frac{x}{2}\right)e^{-x}\cos y$ is the complete solution of (1).

—x—

(4) Show that the equation $\frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2}$ possesses solution of the form

$$\sum_{r=0}^{\infty} c_r e^{-kt}\cos(\alpha_r x+\varepsilon_r)\cos(w_r t+\varepsilon_r)$$

where $c_r, \alpha_r, \varepsilon_r, \delta_r$ are constants and $w_r^2 = \alpha_r^2 c^2 - k^2$

Soln:

Given equation can be written as

$$(D'^2 + 2kD' - c^2D^2)y = 0 \quad \text{—} (1)$$

Let $y = e^{ax+bt}$ be the solution of (1) and

$a^2+b^2=0$

Also, from (1)

$$b^2 + 2kb - c^2a^2 = 0$$

$$b = \frac{-2k \pm \sqrt{4k^2+4a^2c^2}}{2}$$

$$= -k \pm \sqrt{k^2+a^2c^2}$$

In general

$$b_r = -k \pm \sqrt{k^2 + a_r^2 c^2}$$

If $a_r^2 = -\alpha_r^2$

$$b_r = -k \pm \sqrt{k^2 - \alpha_r^2 c^2}$$

$$= -k \pm \sqrt{-(\alpha_r^2 c^2 - k^2)}$$

$$b_r = -k \pm i w_r \quad , \quad \text{where } w_r^2 = \alpha_r^2 c^2 - k^2$$

$\therefore$ from $\circledast$

$$y = e^{ax} e^{bt}$$

$$= e^{\pm i\alpha_r x} e^{(-k \pm i w_r)t}$$

$$= e^{-kt} e^{\pm i\alpha_r x} e^{\pm i w_r t}$$

$$\therefore y = \sum_{r=0}^{\infty} C_r e^{-kt} \cos(\alpha_r x + \varepsilon_r) \cos(w_r t + \varepsilon_r)$$

— x —

(5) Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

<u>Soln</u>.

$$(D^2 - DD')z = \sin x \cos 2y$$

$$(D^2 - DD')z = \frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$$

A.E $\quad m^2 - m = 0$

$m(m-1) = 0$

$m_1 = 0, \quad m_2 = 1$

$C.F = f_1(y) + f_2(y+x)$

$P.I_1 = \frac{1}{2} \frac{1}{D^2 - DD'} \sin(x+2y) \qquad a = 1, b = 2$

$= \frac{1}{2}\left(\frac{1}{-1+2} \sin(x+2y)\right)$

$= \frac{1}{2} \sin(x+2y)$

$P.I_2 = \frac{1}{2} \frac{1}{D^2 - DD'} \sin(x-2y)$

$-a^2 - ab$

$= \frac{1}{2}\left(\frac{1}{-1-2} \sin(x-2y)\right)$

$= \frac{-1}{6} \sin(x-2y)$

$\therefore$ the complete solution is

$z = f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x-2y) - \frac{1}{6}\sin(x-2y$

— x —

(6) Solve $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$

Soln:

$(D^2 + 2DD' + D'^2) z = x^2 + xy + y^2$

A.E $\quad m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m_1 = m_2 = -1$

$C.F = f_1(y-x) + x f_2(y-x)$

$$P.I = \frac{1}{D^2+2DD'+D'^2}(x^2+xy+y^2)$$

$$= \frac{1}{D^2\left[1+\frac{2D'}{D}+\frac{D'^2}{D^2}\right]}(x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left[1+\frac{2D'}{D}+\frac{D'^2}{D^2}\right]^{-1}(x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left[1-\left(\frac{2D'}{D}+\frac{D'^2}{D^2}\right)+\frac{4D'^2}{D^2}\right](x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left[x^2+xy+y^2-\frac{2}{D}(x+2y)-\frac{1}{D^2}(2)+\frac{4}{D^2}(2)\right]$$

$$= \frac{1}{D^2}\left[x^2+xy+y^2-x^2-4xy-x^2+4x^2\right]$$

$$= \frac{1}{D^2}\left[3x^2-3xy+y^2\right]$$

$$= \frac{x^4}{4}-\frac{x^3y}{2}+\frac{x^2y^2}{2}$$

$$= \frac{x^2}{2}\left[\frac{x^2}{2}-\frac{xy}{1}+y^2\right]$$

$$P.I = \frac{x^2}{2}\left[\frac{x^2}{2}+y^2-xy\right]$$

The complete solution is

$$z = f_1(y-x)+xf_2(y-x)+\frac{x^2}{2}\left[\frac{x^2}{2}+y^2-xy\right]$$

— x —

Solve $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial^2 y \partial x} - 6\frac{\partial^3 z}{\partial y^3} = \sin(x+2y) + e^{2x+y}$

Soln:

$(D^3 - 7D'^2 D - 6D'^3) z = \sin(x+2y) + e^{2x+y}$

$(D^3 - 7DD'^2 - 6D'^3) z = \sin(x+2y) + e^{2x+y}$

A.E

$m^3 - 7m - 6 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$\therefore m_1 = -1, \; m_2 = 3, \; m_3 = -2$

$C.F = f_1(y-x) + f_2(y+3x) + f_3(y-2x)$

$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x+2y)$ $\quad a=1, b=2$

$= \frac{1}{-D + 14D' + 24D'} \sin(x+2y)$

$= \frac{1}{38D' - D} \sin(x+2y)$

$= \frac{38D' + D}{1444D'^2 - D^2} \sin(x+2y)$

$= \frac{(38D' + D) \sin(x+2y)}{-5776 + 1}$

$$P.I_1 = -\frac{1}{5775}\left[76\cos(x+2y) + \cos(x+2y)\right]$$

$$= \frac{-77}{5775}\cos(x+2y)$$

$$= \frac{-11}{825}\cos(x+2y)$$

$$= \frac{-1}{75}\cos(x+2y)$$

$$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{2x+y}$$

$$= \frac{1}{8 - 7(2)(1) - 6} e^{2x+y}$$

$$P.I_2 = -\frac{1}{12} e^{2x+y}$$

∴ The complete solution is

$$z = C.F + P.I_1 + P.I_2$$

$$= f_1(y-x) + f_2(y+3x) + f_3(y-2x) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}.$$

— x —

Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$.

Soln:

$$(D^3 - 3D^2D' + 4D'^3)\, z = e^{x+2y}$$

A.E $\quad m^3 - 3m^2 + 4 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$\therefore m_1 = -1,\ m_2 = 2,\ m_3 = 2$

$\therefore$ C.F $= f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$

$$P.I = \frac{1}{D^3 - 3D^2D' + 4D'^3}\, e^{x+2y}$$

$$= \frac{1}{1 - 3(1)(2) + 4(2)^3}\, e^{x+2y}$$

$$= \frac{1}{1-6+32}\, e^{x+2y}$$

$$= \frac{1}{27}\, e^{x+2y}$$

$\therefore$ the complete solution is

$z =$ C.F + P.I

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

— x —

(9) $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y)$

Soln:

A.E $\quad 2m^2 - 5m + 2 = 0$

$$2m^2 - 4m - m + 2 = 0$$

$$2m(m-2) - 1(m-2) = 0$$

$$(2m-1)(m-2) = 0$$

$$m_1 = \frac{1}{2}, \quad m_2 = 2$$

$$C.F = f_1(y + x/2) + f_2(y + 2x)$$

$$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5\sin(2x+y)$$

$$= 5 \frac{x \sin(2x+y)}{4D - 5D'}$$

$$= 5x \frac{4D + 5D'}{16D^2 - 25D'^2} \sin(2x+y)$$

$$= \frac{5x(4D + 5D')}{-64 + 25} \sin(2x+y)$$

$$= \frac{5x}{-39}(4D + 5D')(\sin(2x+y))$$

$$= -\frac{5x}{39}[8\cos(2x+y) + 5\cos(2x+y)]$$

$$= -\frac{5x}{39}[13\cos(2x+y)]$$

$$= \frac{-5x}{3}\cos(2x+y)$$

$\therefore$ the complete solution is

$$z = f_1(y + x/2) + f_2(y + 2x) - \frac{5}{3}x\cos(2x+y)$$

—x—

(5) $$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x+y$$

<u>Soln:</u> $(D^2 + 3DD' + 2D'^2) z = x+y$

A.E $\quad m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m_1 = -1, \; m_2 = -2$

$C.F = f_1(y-x) + f_2(y-2x)$

$$P.I = \frac{1}{D^2 + 3DD' + 2D'^2}(x+y)$$

$$= \frac{1}{D^2\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]}(x+y)$$

$$= \frac{1}{D^2}\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right](x+y)$$

$$= \frac{1}{D^2}\left[x+y - \frac{3}{D}(1) + \frac{2}{D^2}(0)\right]$$

$$= \frac{1}{D^2}\left[x+y-3x\right]$$

$$= \frac{1}{D^2}\left[y-2x\right]$$

$$= \frac{x^2 y}{2} - \frac{x^3}{3}$$

$$P.I = x^2\left[\frac{y}{2} - \frac{x}{3}\right]$$

$\therefore$ The complete solution is

$$z = f_1(y-x) + f_2(y-2x) + x^2\left(y/_2 - \frac{x}{3}\right)$$

Classification of second order partial Differential Equations (PDE)

Defn:

A second order partial differential equation which is linear with respect to the second order partial derivatives, i.e., r, s and t, is said to be a quasi linear PDE of second order.

For example, the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{---} (1)$$

Here $f(x, y, z, p, q)$ need not be linear, is a quasi-linear partial differential equation. Here the coefficients R, S, T are functions of x and y but we assume them to be constant.

The equation (1) is said to be

(1) Elliptic if $S^2 - 4RT < 0$

(2) parabolic if $S^2 - 4RT = 0$ and

(3) Hyperbolic if $S^2 - 4RT > 0$ at a point (x_0, y_0)

If the number of independent variables is two or three, a transformation can always be found to reduce the given PDE to a cannonical form (Normal form)

In general, when the number of independent variables is greater than three, it is not always possible to find such a transformation except in certain special cases.

The idea of reducing the given PDE to a canonical form is that the transformation equation assumes a simple form so that the subsequent analysis of solving the equation is made easy.

<u>cannonical forms</u>

In order to reduce the PDE

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{—} \quad (1)$$

to a cannonical form we apply the transformation

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

such that the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

$$= \xi_x \eta_y - \xi_y \eta_x \neq 0$$

in the domain Ω, where the eqn ① holds.

Now, we have

$$p = \frac{\partial z}{\partial x}$$

ξ η

$$= \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$= \xi_x z_\xi + \eta_x z_\eta$$

$$q = \frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial y}$$

$$= \xi_y z_\xi + \eta_y z_\eta$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left[\xi_x z_\xi + \eta_x z_\eta\right]$$

$$= \xi_x \left[\frac{\partial z_\xi}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial z_\xi}{\partial \eta} \frac{\partial \eta}{\partial x}\right] + z_\xi \xi_{xx}$$

$$+ \eta_x \left[\frac{\partial z_\eta}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z_\eta}{\partial \eta} \frac{\partial \eta}{\partial x}\right] + z_\eta \eta_{xx}$$

$$= \xi_x^2 z_{\xi\xi} + \xi_x \eta_x z_{\xi\eta} + z_\xi \xi_{xx}$$

$$+ \xi_x \eta_x z_{\eta\xi} + \eta_x^2 z_{\eta\eta} + z_\eta \eta_{xx}$$

$$= \xi_x^2 z_{\xi\xi} + z_\xi \xi_{xx} + \eta_x^2 z_{\eta\eta} + z_\eta \eta_{xx}$$

$$+ 2\xi_x \eta_x z_{\xi\eta}.$$

$$s = \frac{\partial^2 z}{\partial x \partial y}$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[z_\xi \xi_x + z_\eta \eta_x\right]$$

$$= \xi_x\left[\frac{\partial z_\xi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z_\xi}{\partial \eta}\frac{\partial \eta}{\partial y}\right] + z_\xi \xi_{xy}$$

$$+ \eta_x\left[\frac{\partial z_\eta}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z_\eta}{\partial \eta}\frac{\partial \eta}{\partial y}\right] + z_\eta \eta_{xy}$$

$$= \xi_x z_{\xi\xi}\xi_y + z_{\xi\eta}\eta_y + z_\xi \xi_{xy}$$

$$+ \eta_x z_{\eta\xi}\xi_y + z_{\eta\eta}\eta_y\eta_x + z_\eta \eta_{xy}$$

$$= z_{\xi\xi}\xi_x\xi_y + z_{\eta\eta}\eta_x\eta_y + z_{\xi\eta}\xi_x\eta_y$$

$$+ z_{\xi\eta}\eta_x\xi_y + z_\eta \eta_x\eta_y + z_\xi \xi_x\xi_y.$$

$1 \times \frac{1}{0} = \infty$

$$t = \frac{\partial^2 z}{\partial y^2}$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left[z_\xi \xi_y + z_\eta \eta_y\right]$$

$$= \xi_y\left[\frac{\partial z_\xi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z_\xi}{\partial \eta}\frac{\partial \eta}{\partial y}\right] + z_\xi \xi_{yy}$$

$$+\eta_y\left[\frac{\partial z_\eta}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial z_\eta}{\partial \eta}\frac{\partial \eta}{\partial y}\right]+z_\eta\,\eta_{yy}$$

$$=\xi_y\xi_y\,z_{\xi\xi}+\xi_y\eta_y\,z_{\xi\eta}+z_\xi\,\xi_{yy}$$

$$+\eta_y\xi_y\,z_{\eta\xi}+\eta_y\eta_y\,z_{\eta\eta}+z_\eta\,\eta_{yy}$$

$$=\xi_y^2\,z_{\xi\xi}+z_\xi\,\xi_{yy}+\eta_y^2\,z_{\eta\eta}+z_\eta\,\eta_{yy}$$

$$+2\xi_y\eta_y\,z_{\xi\eta}.$$

$\xi_y\,\xi_y$

Substituting the values of p, q, r, s and t in (1) we get the equation of the form.

$$A(\xi_x,\xi_y)\,z_{\xi\xi}+2B(\xi_x,\xi_y,\eta_x,\eta_y)\,z_{\xi\eta}+$$

$$A(\eta_x,\eta_y)\,z_{\eta\eta}=F(\xi,\eta,z,z_\xi,z_\eta)$$

Case (i): $S^2-4RT>0$.

Then the equation $R\lambda^2+S\lambda+T=0$ has real and distinct roots.

$\lambda_1 \quad \lambda_2$

Let these roots be λ_1 and λ_2.

We choose ξ and η such that

$$\xi_x=\lambda_1\xi_y,\quad \eta_x=\lambda_2\eta_y.$$

$$\xi_x-\lambda_1\xi_y=0$$

S.E $\frac{dx}{1} = \frac{dy}{-\lambda} = \frac{d\xi}{0}$

$\frac{dx}{1} = \frac{d\xi}{0}$

$d\xi = 0$

$\Rightarrow \xi = \text{constant}$

$\frac{dx}{1} = \frac{dy}{-\lambda_1}$

$\Rightarrow -\lambda_1 dx = dy$

$\Rightarrow \frac{dy}{dx} = -\lambda_1$

$\Rightarrow \frac{dy}{dx} + \lambda_1(x,y) = 0$

$|||^{ly}$, considering

$\eta_x - \lambda_2 \eta_y = 0$

$\frac{dy}{dx} + \lambda_2(x,y) = 0$

Let the solution of these equations be given by $f_1(x,y) = \text{constant}$ and $f_2(x,y) = \text{const}$

Thus we get $\xi = f_1(x,y)$ and $\eta = f_2(x,y)$

case (ii)

If $S^2 - 4RT = 0$

Then the equation $R\lambda^2 + S\lambda + T = 0$ will have equal roots say $\lambda_1 = \lambda_2 = \lambda$

We choose $\xi = f_1(x, y)$

Which is a constant and is the solution of

$$\frac{dy}{dx} + \lambda(x, y) = 0$$

In this case η will be independent of ξ.

<u>case(iii)</u>

$$S^2 - 4RT < 0$$

Then the equation $R\lambda^2 + S\lambda + T = 0$ will have imaginary roots and hence ξ and η will be complex.

Let $\xi = \alpha + i\beta$ and
$\eta = \alpha - i\beta$, where α, β are real

$Z_\alpha = \frac{\partial z}{\partial \alpha}$

We have

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2i}$$

$\cancel{Z_{\alpha\alpha} = \frac{\partial^2 Z_\alpha}{\partial \alpha \partial \alpha}}$

Proceeding as in the case (i), we get

$$Z_{\alpha\alpha} + Z_{\beta\beta} = \phi(\alpha, \beta, z, Z_\alpha, Z_\beta)$$

Which is the required canonical form of the elliptic partial differential equation.

— ⟶ —

Problems

(1) Reduce the PDE $y^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} = \frac{y^2}{2}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}$.

to canonical form and hence solve it. —(1)

Soln:

In the given eqn.

$R = y^2, \quad S = -2xy, \quad T = x^2$

Now, $S^2 - 4RT = \cancel{4x^2y^2} - \cancel{4x^2y^2} = 0$

$S^2 - 4RT = 0$

∴ Given equation is a parabolic differential equation.

∴ The equation $R\lambda^2 + S\lambda + T = 0$ has equal roots

$$y^2\lambda^2 - 2xy\lambda + x^2 = 0$$

$$(y\lambda - x)^2 = 0$$

$$\Rightarrow y\lambda - x = 0$$

$$\Rightarrow \lambda = \frac{x}{y}$$

∴ We have $\frac{dy}{dx} + \lambda(x, y) = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{y} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow x\,dx + y\,dy = 0$$

Integrating, $\frac{x^2}{2}+\frac{y^2}{2}=\frac{c}{2}$

$$\Rightarrow x^2+y^2=\text{constant}$$

$$\therefore \xi = x^2+y^2$$

$$\eta = x^2-y^2 \quad (\because \eta \text{ is independent of } \xi)$$

$\xi = x^2+y^2$	$\eta = x^2-y^2$
$\xi_x = 2x$	$\eta_x = 2x$
$\xi_y = 2y$	$\eta_y = -2y$
$\xi_{xx} = 2$	$\eta_{xx} = 2$
$\xi_{yy} = 2$	$\eta_{yy} = -2$
$\xi_{xy} = 0$	$\eta_{xy} = 0$.

$$P = Z_\xi \xi_x + Z_\eta \eta_x$$
$$= Z_\xi 2x + Z_\eta 2x$$
$$= 2x(Z_\xi + Z_\eta)$$

$$q = Z_\xi \xi_y + Z_\eta \eta_y$$
$$= Z_\xi 2y + Z_\eta(-2y)$$
$$= 2y(Z_\xi - Z_\eta)$$

$$r = \xi_x^2 Z_{\xi\xi} + \xi_{xx} Z_\xi + 2\xi_x \eta_x Z_{\xi\eta} + \eta_x^2 Z_{\eta\eta} + Z_\eta \eta_{xx}$$
$$= 4x^2 Z_{\xi\xi} + 2Z_\xi + 8x^2 Z_{\xi\eta} + 4x^2 Z_{\eta\eta} + 2Z_\eta.$$

$t = \xi_y^2 z_{\xi\xi} + \xi_{yy} z_\xi + 2\xi_y \eta_y z_{\xi\eta} + \eta_y^2 z_{\eta\eta} + z_\eta \eta_{yy}$

$= 4y^2 z_{\xi\xi} + 2z_\xi - 8y^2 z_{\xi\eta} + 4y^2 z_{\eta\eta} - 2z_\eta.$

$s = z_{\xi\xi} \xi_x \xi_y + z_{\eta\eta} \eta_x \eta_y + z_{\xi\eta} \eta_x \xi_y + z_{\xi\eta} \xi_x \eta_y$

$+ z_\eta \eta_{xy} + z_\xi \xi_{xy}.$

$= 4xy z_{\xi\xi} - 4xy z_{\eta\eta} + 4xy z_{\xi\eta} - 4xy z_{\xi\eta}$

$- 4xy z_\eta + 4xy z_\xi.$

Substitute p, q, r, s, t in (1)

$y^2[4x^2 z_{\xi\xi} + 2z_\xi + 8x^2 z_{\xi\eta} + 4x^2 z_{\eta\eta} + 2z_\eta]$

$-2xy[4xy z_{\xi\xi} - 4xy z_{\eta\eta} + 4xy z_{\xi\eta} - 4xy z_{\eta\xi}]$

$+ x^2[4y^2 z_{\xi\xi} + 2z_\xi - 8y^2 z_{\xi\eta} + 4y^2 z_{\eta\eta} - 2z_\eta]$

$= \frac{y^2}{x}(2x z_\xi + 2x z_\eta) + \frac{x^2}{y}(2y z_\xi - 2y z_\eta)$

$16x^2y^2 z_{\eta\eta} = 0$

$\Rightarrow z_{\eta\eta} = 0$

$\frac{\partial^2 z}{\partial \eta^2} = 0.$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \eta}\right)=0$$

$$\frac{\partial z}{\partial \eta}=A$$

$$z=A\eta+B.$$

$$z=A(\xi)\eta+B(\xi)$$

$$\therefore z=A(x^2+y^2)\eta+B(x^2+y^2)$$

$$z=A(x^2+y^2)(x^2-y^2)+B(x^2+y^2)$$

which is the required solution of the given equation.

(2) Reduce the PDE $(n-1)^2\frac{\partial^2 z}{\partial x^2}-y^{2n}\frac{\partial^2 z}{\partial y^2}=ny^{2n-1}\frac{\partial z}{\partial y}$. to canonical form and find its general solution.

Solution:

This is of the form $Rr+Ss+Tt+f(x,y,z,p,q)=0$

$R=(n-1)^2$, $S=0$, $T=-y^{2n}$.

$$S^2-4RT=0-4(n-1)^2(-y^{2n})$$

$$=4(n-1)^2y^{2n}>0$$

$\therefore$ Given PDE is a hyperbolic

$\therefore$ The equation $R\lambda^2+S\lambda+T=0$ will have real and unequal roots.

Let them be λ_1, λ_2

$$(n-1)^2\lambda^2 - y^{2n} = 0$$

$$(n-1)^2\lambda^2 = y^{2n}$$

$$\lambda^2 = \frac{y^{2n}}{(n-1)^2}$$

$$\lambda = \pm\frac{y^n}{n-1}$$

i.e., $\lambda_1 = \frac{y^n}{n-1}$, $\lambda_2 = -\frac{y^n}{n-1}$

$\therefore$ We have $\frac{dy}{dx} + \frac{y^n}{n-1} = 0$

$$\frac{dy}{dx} = \frac{-y^n}{(n-1)}$$

$$\frac{dy}{y^n} = \frac{dx}{(n-1)}$$

$$\Rightarrow (n-1)y^{-n}dy = dx$$

Integrating $(n-1)\frac{y^{-n+1}}{(1-n)} = -x+c$

$$x - y^{-n+1} = c$$

$|||^{ly}$, taking $\frac{dy}{dx} - \frac{y^n}{n-1} = 0$

We have $x + y^{-n+1} = c$

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Let $\xi = x + y^{1-n}$

$\eta = x - y^{1-n}$

$\xi = x + y^{1-n}$

$\xi_x = 1$

$\xi_y = (1-n)\, y^{-n}$

$\xi_{xx} = 0$

$\xi_{yy} = -n(1-n)\, y^{-n-1}$

$= (n^2 - n)\, y^{-n-1}$

$\xi_{xy} = 0$

$\eta = x - y^{1-n}$

$\eta_x = 1$

$\eta_y = -(1-n)\, y^{-n}$

$\eta_{xx} = 0$

$\eta_{yy} = n(1-n)\, y^{-n-1}$

$= (n - n^2)\, y^{-n-1}$

$\eta_{xy} = 0$

$p = z_\xi \xi_x + z_\eta \eta_x$

$= z_\xi + z_\eta$

$q = z_\xi \xi_y + z_\eta \eta_y$

$= z_\xi (1-n) y^{-n} - z_\eta (1-n) y^{-n}$

$= (1-n) y^{-n} (z_\xi - z_\eta)$

$r = \xi_x^2 z_{\xi\xi} + \xi_{xx} z_\xi + 2 \xi_x \eta_x z_{\xi\eta} + \eta_x^2 z_{\eta\eta} + z_\eta \eta_{xx}$

$= z_{\xi\xi} + 0 + 2 z_{\xi\eta} + z_{\eta\eta} + 0$

$= z_{\xi\xi} + 2 z_{\xi\eta} + z_{\eta\eta}.$

$$t = \xi_y^2 z_{\xi\xi} + \xi_{yy} z_\xi + 2\xi_y \eta_y + \eta_y^2 z_{\eta\eta} + z_\eta \eta_{yy}$$

$$= (1-n)^2 y^{-2n} z_{\xi\xi} + (n^2-n) y^{-n-1} z_\xi - 2(1-n)^2 y^{-2n} z_{\xi\eta}$$

$$+ (1-n)^2 y^{-2n} z_{\eta\eta} + (n-n^2) y^{-n-1} z_\eta$$

$$s = z_{\xi\xi} \xi_x \xi_y + z_{\eta\eta} \eta_x \eta_y + z_{\xi\eta} \eta_x \xi_y + z_{\eta\xi} \xi_x \eta_y$$

$$+ z_\eta \eta_{xy} + z_\xi \xi_x \xi_y$$

$$= (1-n) y^{-n} z_{\xi\xi} - (1-n) y^{-n} z_{\eta\eta} + (1-n) y^{-n} z_{\xi\eta}$$

$$-(1-n) y^{-n} z_{\eta\xi} + 0 + 0$$

$$s = (1-n) y^{-n} z_{\xi\xi} - (1-n) y^{-n} z_{\eta\eta}.$$

$$(n-1)^2 \left[z_{\xi\xi} + 2 z_{\xi\eta} + z_{\eta\eta} \right] - y^{2n} \left[(1-n)^2 y^{-2n} z_{\xi\xi} \right.$$

$$+ (n^2-n) y^{-n-1} z_\xi - 2(1-n)^2 y^{-2n} z_{\xi\eta} +$$

$$\left. (1-n)^2 y^{-2n} z_{\eta\eta} + (n-n^2) y^{-n-1} z_\eta \right]$$

$$\cancel{(n-1)^2 z_{\xi\xi}} + 2(n-1)^2 z_{\xi\eta} + \cancel{(n-1)^2 z_{\eta\eta}} - \cancel{(1-n)^2 z_{\xi\xi}}$$

$$-(n^2-n) y^{2n-n-1} z_\xi + 2(1-n^2) z_{\xi\eta} - \cancel{(1-n)^2 z_{\eta\eta}}$$

$$-(n-n^2) y^{2n-n-1} z_\eta = y^{2n-n-1} (n-n^2) z_\xi$$

$$- y^{2n-n-1} (n-n^2) z_\eta$$

$$\Rightarrow 4(n-1)^2 z_{\xi\eta} = 0$$

$$\Rightarrow z_{\xi\eta} = 0$$

i.e., $\dfrac{\partial^2 z}{\partial \xi \eta^2} = 0$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial \eta}\right) = 0$$

$$\therefore \frac{\partial z}{\partial \eta} = A$$

Integrating, $z = A\eta + B$

$$z = A(\xi)\eta + B(\xi)$$

$$z = A(x + y^{1-n})\eta + B(x - y^{1-n})$$

— x —

Reduce the PDE

1) $\dfrac{\partial^2 z}{\partial x^2} + x^2 \dfrac{\partial^2 z}{\partial y^2} = 0$

2) $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

Adjoint operators

Let $Lu = \phi$ —(1)

where L is the differential operator given by

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + a_n(x) \quad \text{—(2)}$$

The purpose of introducing the adjoint differential operator L^* associated with L is to form the product vLu and integrate it over the interval of interest.

Let $$\int_A^B vLu\,dx = [\quad]_A^B + \int_A^B uL^*v\,dx \quad \text{—(3)}$$

which is obtained after repeated integration by parts. Here L^* is the operator adjoint to L, where the functions u and v are completely arbitrary except that Lu and L^*v should exist.

Defn:

If the operator $L = L^*$, then L is called a self adjoint operator.

(1) If L is the operator

$$R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x\partial y} + T\frac{\partial^2}{\partial y^2} + P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + Z \quad \text{—(1)}$$

and M is the adjoint operator defined by

$$M_w = \frac{\partial^2}{\partial x^2}(Rw) + \frac{\partial^2}{\partial x \partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) + \frac{\partial}{\partial x}(Pw) + \frac{\partial}{\partial y}(Qw)$$

$$+ Zw \quad \text{—— (2)}$$

then show that

$$\iint_S (wLz - zM_w)\, dx\,dy = \int_C [u\cos(n,x) + v\cos(n,y)]\, ds$$

where C is the closed curve enclosing an area S and

$$u = Rw\frac{\partial z}{\partial x} - z\frac{\partial}{\partial x}(Rw) - z\frac{\partial}{\partial y}(Sw) + Pzw \quad \text{—— (3)}$$

$$v = Sw\frac{\partial z}{\partial x} + Tw\frac{\partial z}{\partial x} - z\frac{\partial}{\partial y}(Tw) + Qzw \quad \text{—— (4)}$$

If $R_x + \frac{1}{2}S_y = P$ —— (5)

$\frac{1}{2}S_x + T_y = Q$ —— (6)

show that the operator L is self adjoint

Solution

Let $L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} + P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + Z$

$$wLz = wR\frac{\partial^2 z}{\partial x^2} + wS\frac{\partial^2 z}{\partial x \partial y} + wT\frac{\partial^2 z}{\partial y^2} + wP\frac{\partial z}{\partial x} + wQ\frac{\partial z}{\partial y} + zZw.$$

$$zM_w = z\frac{\partial^2}{\partial x^2}(Rw) + z\frac{\partial^2}{\partial x \partial y}(Sw) + z\frac{\partial^2}{\partial y^2}(Tw) + z\frac{\partial}{\partial x}(Pw)$$

$$+ z\frac{\partial}{\partial y}(Qw) + zZw$$

$$wLz - zMw = \left[wR\frac{\partial^2 z}{\partial x^2} - z\frac{\partial^2 (Rw)}{\partial x^2}\right] + \left[Sw\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2}{\partial x \partial y}(Sw)\right] + \left[Tw\frac{\partial^2 z}{\partial y^2} - z\frac{\partial^2 (Tw)}{\partial y^2}\right]$$

$$+ Pw\frac{\partial z}{\partial x} + z\frac{\partial}{\partial x}(Pw) + Qw\frac{\partial z}{\partial y} + z\frac{\partial}{\partial y}(Qw)$$

$$= \frac{\partial}{\partial x}\left(wR\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left[z\frac{\partial (Rw)}{\partial x}\right] + \frac{\partial}{\partial y}\left(Sw\frac{\partial z}{\partial x}\right)$$

$$- \frac{\partial}{\partial x}\left(z\frac{\partial (Sw)}{\partial y}\right) + \frac{\partial}{\partial y}\left(Tw\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left[z\frac{\partial (Tw)}{\partial y}\right]$$

$$+ \frac{\partial}{\partial x}(Pwz) + \frac{\partial}{\partial y}(Qwz)$$

$$= \frac{\partial}{\partial x}\left\{wR\frac{\partial z}{\partial x} - z\frac{\partial (Rw)}{\partial x} - z\frac{\partial (Sw)}{\partial y} + Pwz\right\}$$

$$+ \frac{\partial}{\partial y}\left\{Sw\frac{\partial z}{\partial x} + Tw\frac{\partial z}{\partial y} - z\frac{\partial}{\partial y}(Tw) + Qwz\right\}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

where u and v are given by (3) and (4) respectively.

$$\therefore \iint_S (wLz - zMw)\,dx\,dy = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy$$

$$= \int_C (lu + mv)\,ds$$

$$= \int_C \{u\cos(n,x) + v\cos(n,y)\}\,ds$$

which is required.

$$Mw = \frac{\partial^2}{\partial x^2}(Rw) + \frac{\partial^2}{\partial x \partial y}(Sw) + \frac{\partial^2}{\partial y^2}(Tw) - \frac{\partial}{\partial x}(Pw)$$

$$- \frac{\partial}{\partial y}(Qw) + Zw.$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(Rw)\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(Sw)\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(Tw)\right)$$

$$- P\frac{\partial w}{\partial x} - Q\frac{\partial w}{\partial y} + Zw$$

$$= \frac{\partial}{\partial x}\left[R\frac{\partial w}{\partial x} + w\frac{\partial R}{\partial x}\right] + \frac{\partial}{\partial x}\left[S\frac{\partial w}{\partial y} + w\frac{\partial S}{\partial y}\right]$$

$$+ \frac{\partial}{\partial y}\left[T\frac{\partial w}{\partial y} + w\frac{\partial T}{\partial y}\right] - Pw_x - wP_x - Qw_y - wQ_y +$$

$$Zw$$

$$= Rw_{xx} + 2R_x w_x + R_{xx}w + Sw_{xy} + S_y w_x$$

$$+ S_x w_y + wS_{xy} + Tw_{yy} + 2T_y w_y + wT_{yy}$$

$$- Pw_x - wP_x - Q_y w - Qw_y + Zw.$$

$$= Rw_{xx} + (2R_x + S_y)w_x + Sw_{xy} + Tw_{yy} - Pw_x$$

$$- wP_x - Q_y w - Qw_y + (2T_y + S_x)w_y$$

$$+ R_{xx}w + wS_{xy} + wT_{yy} + Zw.$$

$$= Rw_{xx} + 2Pw_x + Sw_{xy} + Tw_{yy} - Pw_x - wP_x$$

$$- Q_y w - Qw_y + 2Qw_y + R_{xx}w + wS_{xy}$$

$$+ wT_{yy} + Zw.$$

$= R w_{xx} + P w_x + S w_{xy} + T w_{yy} + P w_x - w P_x$

$- Q_y w + Q w_y + R_{xx} w + w S_{xy} + w T_{yy} + Z w.$

$$= R\frac{\partial^2 w}{\partial x^2} + S\frac{\partial^2 w}{\partial x \partial y} + T\frac{\partial^2 w}{\partial y^2} + P\frac{\partial w}{\partial x} + Q\frac{\partial w}{\partial y} + Zw$$

$$+ w\frac{\partial^2 R}{\partial x^2} + w\frac{\partial^2 S}{\partial x \partial y} + w\frac{\partial^2 T}{\partial y^2} - w\frac{\partial P}{\partial x} - w\frac{\partial Q}{\partial y}.$$

$\therefore L = L^*$

Hence the operator is self adjoint.

(2) Construct an operator adjoint to the laplace operator given by

$$L(u) = u_{xx} + u_{yy}. \quad (u_{xx} + u_{yy} = 0 \text{ Laplace eqn})$$

Verify whether it is self adjoint.

Solution

$$L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} + P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + Z$$

Given $Lu = u_{xx} + u_{yy}$.

$R = 1, \quad S = 0, \quad T = 1, \quad P = 0, \quad Q = 0, \quad Z = 0$

We know that

$$Mw = L^* = \frac{\partial^2 (Rw)}{\partial x^2} + \frac{\partial^2 (Sw)}{\partial x \partial y} + \frac{\partial^2 (Tw)}{\partial y^2} - \frac{\partial}{\partial x}(Pw) - \frac{\partial}{\partial y}(Qw) + Zw.$$

$$\therefore L^* = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}$$

$$L^* = W_{xx} + W_{yy}$$

$$\therefore L = L^*$$

$\therefore$ L is self adjoint.

Hence the Laplace operator is a self adjoint operator.

(3) show that $Lu = c^2 u_{xx} - u_{tt}$ is self adjoint.

Soln:

$$L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} + P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + Z.$$

Given $Lu = c^2 u_{xx} - u_{tt}$.

$R = c^2$, $S = 0$, $t = -1$, $P = 0$, $Q = 0$, $Z = 0$.

$$Mw = L^* = \frac{\partial^2 (RW)}{\partial x^2} + \frac{\partial^2 (SW)}{\partial x \partial y} + \frac{\partial^2 (TW)}{\partial y^2} - \frac{\partial (PW)}{\partial x} - \frac{\partial}{\partial y}(QW) + ZW$$

$$= \frac{\partial^2 (c^2 W)}{\partial x^2} + 0 + \frac{\partial^2 (-W)}{\partial t^2}$$

$$= c^2 \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2}{\partial t^2}(-W)$$

$$= c^2 \frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial t^2}$$

$$= c^2 W_{xx} - W_{tt}$$

$\therefore L^* = L$

Hence L is self adjoint.

(8) verify whether $Lu = u_{xx} - u_t$ is self adjoint

Solution

$$L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} + P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + Z$$

Given $Lu = u_{xx} - u_t$.

$R = 1,\ S = 0,\ T = 0,\ P = 0,\ Q = -1,\ Z = 0.$

$$Mw = L^* = \frac{\partial^2 (RW)}{\partial x^2} + \frac{\partial^2 (SW)}{\partial x \partial y} + \frac{\partial^2 (TW)}{\partial y^2} - \frac{\partial}{\partial x}(PW) - \frac{\partial}{\partial y}(QW) + ZW$$

$$= \frac{\partial^2 W}{\partial x^2} + 0 + 0 + 0 - \frac{\partial (-W)}{\partial t} + 0$$

$$= \frac{\partial^2 W}{\partial x^2} + \frac{\partial W}{\partial t}$$

$$L^* = W_{xx} + W_t$$

$\therefore L^* \neq L$

Hence L is not self adjoint

— x —

Riemann's Method

Any linear hyperbolic partial differential equation can be written in the form

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = f(x,y) \quad \text{--- (1)}$$

where a, b, c are functions of x and y.

$$Lz = f(x,y)$$

where $$L = \frac{\partial^2}{\partial x \partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c \quad \text{--- (2)}$$

We know that

$$Mw = \frac{\partial^2 (Rw)}{\partial x^2} + \frac{\partial^2 (Sw)}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}(Tw) - \frac{\partial}{\partial x}(Pw) - \frac{\partial}{\partial y}(Qw) + Zw$$

From (2), $R=0$, $S=1$, $T=0$, $P=a$, $Q=b$, $Z=c$.

$$\therefore Mw = 0 + \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial (aw)}{\partial x} - \frac{\partial}{\partial y}(bw) + cw \quad \text{--- (3)}$$

Now,

$$wLz = w\frac{\partial^2 z}{\partial x \partial y} + aw\frac{\partial z}{\partial x} + bw\frac{\partial z}{\partial y} + wcz \quad \text{--- (4)}$$

$$zMw = z\frac{\partial^2 w}{\partial x \partial y} - z\frac{\partial}{\partial x}(aw) - z\frac{\partial}{\partial y}(bw) + zcw \quad \text{--- (5)}$$

$$wLz - zMw = w\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y} + aw\frac{\partial z}{\partial x} + z\frac{\partial}{\partial x}(aw) + bw\frac{\partial z}{\partial y} + z\frac{\partial}{\partial y}(bw) \quad \text{--- (6)}$$

$$= \frac{\partial}{\partial y}\left(w\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}(awz) + \frac{\partial}{\partial y}(bwz)$$

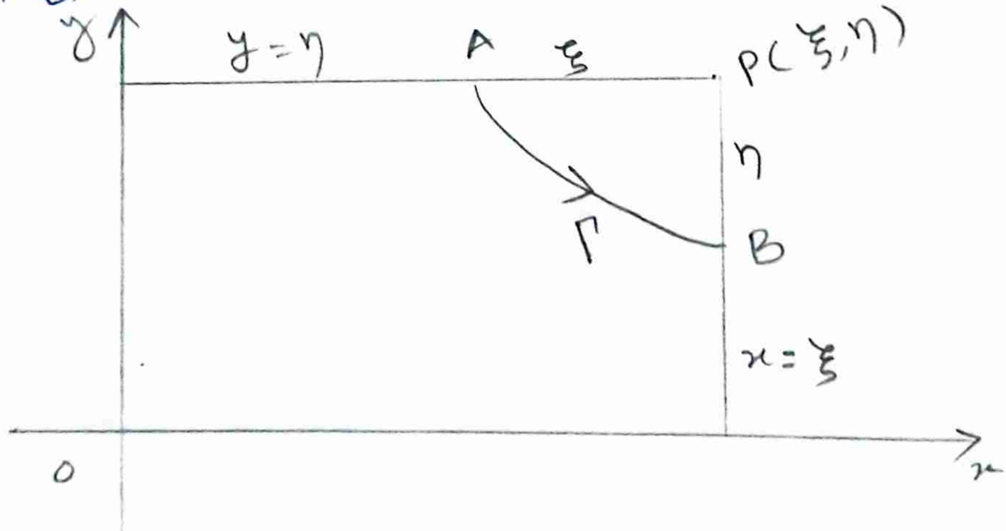
$$= \frac{\partial}{\partial x}\left(awz - z\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(bwz + w\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

where $u = awz - z\frac{\partial w}{\partial y}$

$v = bwz + w\frac{\partial z}{\partial x}$

Now consider an arc AB of a curve Γ. Where PA is parallel to x axis, PB parallel to y axis and $P(\xi, \eta)$ is any point.



Let S denote the areal enclosed by the contour ABPA clearly on AP,

$y = \eta$

$\therefore dy$ is zero, on PB, $x = \xi$

$\therefore dx$ is zero.

Now, $\iint_S (WLZ - ZMW)\,dx\,dy = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\,dy$

By Green's theorem,

$$= \int_C (u\,dy - v\,dx)$$

Where C is the closed contour ABPA.

$$= \int_A^B u\,dy - v\,dx + \int_B^P u\,dy - v\,dx + \int_P^A u\,dy - v\,dx$$

$$= \int_A^B u\,dy - \int_A^B v\,dx + \int_B^P u\,dy - \int_B^P v\,dx + \int_P^A u\,dy - \int_P^A v\,dx$$

$$= \int_A^B u\,dy - \int_A^B v\,dx + \int_B^P u\,dy - \int_P^A v\,dx \quad \text{——— (I)}$$

($\because$ on BP $\cancel{dy}$ $=0$
on AP $dy = 0$)

$$= \int_A^B \left(awz - z\frac{\partial w}{\partial y}\right) dy - \int_A^B \left(bwz + w\frac{\partial z}{\partial x}\right) dx$$

$$+ \int_B^P \left(awz - z\frac{\partial w}{\partial y}\right) dy - \int_P^A \left(bwz + w\frac{\partial z}{\partial x}\right) dx$$

$$\int_P^A v\,dx = \int_P^A \left(bwz + w\frac{\partial z}{\partial x}\right) dx$$

$$= \int_P^A bwz\,dx + \int_P^A w\frac{\partial z}{\partial x}\,dx$$

$$= \int_P^A bwz\,dx + [wz]_P^A - \int_P^A z\,w_x\,dx$$

Let $u = w$, $dv = \frac{\partial w}{\partial x}\,dx$ Let $u = w$,

$dv = \frac{\partial z}{\partial x}\,dx$

$dv = dz$

$v = z$.

$$= [wz]_A - [wz]_P + \int_P^A z[bw - w_x]\,dx \quad \text{— } (*)_1$$

$$\int_A^B u\,dy - v\,dx = \int_A^B \left(awz - z\frac{\partial w}{\partial y}\right)dy - \left(bwz + w\frac{\partial z}{\partial x}\right)dx$$

$$= \int_A^B wz(a\,dy - b\,dx) - \int_A^B w\frac{\partial z}{\partial x}\,dx + z\frac{\partial w}{\partial y}\,dy \quad \text{— } (*_2)$$

Now, $\displaystyle\int_B^P u\,dy = \int_B^P \left(awz - z\frac{\partial w}{\partial y}\right)dy$ — $(*_3)$

Using $(*_1)$, $(*_2)$ and $(*_3)$ in I.

$$= \int_A^B wz(a\,dy - b\,dx) - \int_A^B w\frac{\partial z}{\partial x}\,dx + z\frac{\partial w}{\partial y}\,dy$$

$$+ \int_B^P \left(awz - z\frac{\partial w}{\partial y}\right)dy - [wz]_A + [wz]_P$$

$$- \int_P^A z\left(bw - \frac{\partial w}{\partial x}\right)dx$$

$$[WZ]_P = \iint_S (WLz - zMW)\,dx\,dy - \int_A^B Wz(a\,dy - b\,dx)$$
$$+ \int_A^B W\frac{\partial z}{\partial x}dx + z\frac{\partial W}{\partial y}dy - \int_B^P \left(aWz - z\frac{\partial W}{\partial y}\right)dy$$
$$+ [Wz]_A + \int_P^A z\left(bW - \frac{\partial W}{\partial x}\right)dx \quad \text{--- (II)}$$

The function W is arbitrary

We choose W to satisfy the following conditions

$$\left.\begin{array}{l} \text{(i) } MW = 0 \quad (\text{through } xy \text{ plane}) \\ \text{(ii) } \dfrac{\partial W}{\partial y} = aW \quad \text{on } x = \xi \\ \text{(iii) } \dfrac{\partial W}{\partial x} = bW \quad \text{on } y = \eta. \\ \text{(iv) } [W]_P = 1. \end{array}\right\} \quad \text{III}$$

Using III in II

$$[z]_P = \iint_S WLz\,dx\,dy - \int_A^B Wz(a\,dy - b\,dx) + [Wz]_A$$
$$+ \int_A^B W\frac{\partial z}{\partial x}dx + z\frac{\partial W}{\partial y}dy \quad \text{--- IV}$$

Where $Lz = f(x, y)$

This (iv) gives the value of z at any pt. $P(\xi, \eta)$. when the values of z and $\frac{\partial z}{\partial x}$ are given in the curve AB.

However, if the values of x and $\frac{\partial z}{\partial y}$ are given, then

$$\int_A^B \frac{\partial}{\partial y}(wz)\,dy + \frac{\partial}{\partial x}(wz)\,dx = \int_A^B z\frac{\partial w}{\partial y}\,dy + w\frac{\partial z}{\partial y}\,dy$$

$$+ \int_A^B w\frac{\partial z}{\partial x}\,dx + z\frac{\partial w}{\partial x}\,dx.$$

$$\therefore \int_A^B w\frac{\partial z}{\partial x}\,dx + z\frac{\partial w}{\partial y}\,dy = \int_A^B \frac{\partial}{\partial y}(wz)dy + \frac{\partial}{\partial x}(wz)dx$$

$$- \int_A^B w\frac{\partial z}{\partial y}\,dy + z\frac{\partial w}{\partial x}\,dx$$

$$= \int_A^B d(wz) - \int_A^B w\frac{\partial z}{\partial y}\,dy + z\frac{\partial w}{\partial x}\,dx$$

$$= [wz]_A^B - \int_A^B w\frac{\partial z}{\partial y}\,dy + z\frac{\partial w}{\partial x}\,dx$$

$$= [wz]_B - [wz]_A - \int_A^B w\frac{\partial z}{\partial y}\,dy + z\frac{\partial w}{\partial x}\,dx \quad \text{— (v)}$$

Using (v) in (iv)

$$[z]_P = \iint_S wLz\,dx\,dy - \int_A^B wz(a\,dy - b\,dx) + [wz]_A$$

$$+ [wz]_B - [wz]_A - \int_A^B w\frac{\partial z}{\partial y}\,dy + z\frac{\partial w}{\partial x}\,dx$$

$$= \iint_S wLz\,dx\,dy - \int_A^B wz(a\,dy - b\,dx) + [wz]_B$$

$$- \int_A^B w\frac{\partial z}{\partial y}\,dy + z\frac{\partial w}{\partial x}\,dx \quad \text{— (vi)}$$

Adding $\overline{IV}$ and $\overline{VI}$

$$2[z]_P = 2\iint_S WLz\,dx\,dy - 2\int_A^B Wz(a\,dy - b\,dx) +$$
$$[Wz]_A + [Wz]_B + \int_A^B W\frac{\partial z}{\partial x}dx + z\frac{\partial W}{\partial y}dy$$
$$-\int_A^B W\frac{\partial z}{\partial y}dy + z\frac{\partial W}{\partial x}dx.$$

$\div 2$,

$$[z]_P = \iint_S WLz\,dx\,dy - \int_A^B Wz(a\,dy - b\,dx) + \frac{1}{2}[Wz]_A$$
$$+\frac{1}{2}[Wz]_B + \frac{1}{2}\int_A^B W\frac{\partial z}{\partial x}dx + z\frac{\partial W}{\partial y}dy$$
$$-\frac{1}{2}\int_A^B W\frac{\partial z}{\partial y}dy + z\frac{\partial W}{\partial x}dx \qquad \text{--- } \overline{VII}$$

Equation $\overline{VI}$ is used when z and $\frac{\partial z}{\partial y}$ are given.

Equation $\overline{VII}$ is used when $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are given.

— x —