19PMA09

TOPOLOGY

II M.Sc. MATHEMATICS

III SEMESTER

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TOPOLOGY

Unit - I Topological Spaces

Topological Spaces - Basis for a topology. The order topology - the product topology and X × Y - The Subspace topology - closed sets and limit points

(chapter: 2, sec 12-17)

Unit - II Continous functions

Continous functions - The product of topology - The metric topology.

[chapter 2: Sec 18-21]

Unit - III connectedness:

connected spaces - Connected Subspace of the real line - components and local connetness [chap: 3 Sec 23-25]

Unit - IV Compactness

compact spaces - compact. Subspace of the real line - limit point Compactness - local compactness.

(chapter 3; Sec 26-29)

Unit - V

Countability and Seperation axioms.

The countability axioms - the Seperation axioms - normal spaces - The

Uryshom lemma - The Urysohm metrization theorem - The tietze extension theorem

[cha 4 : sec 30-35]

Text Books :

James . R. munkres - Topology IInd Edi

UNIT- I

Topological spaces - Basis for a topology - The Order topology - the product topology and X × Y - The Subspace topology - Closed sets and Limit points

UNIT - I

TOPOLOGICAL SPACES

DEF : Topology :

Let X be any set τ be the collection of Subset of X, then τ is said to be a topology on X. If it satisfies the following conditions

(i) ϕ and X in τ

(ii) The union of the elements of any sub collections of τ is in τ.

(iii) The intersection of the elements of any finite sub collection of τ is in τ.

DEF : Topological Spaces :

A set X in which a topological τ has been specified is called a topological spaces (X, τ). (OR)

The topological spaces is an order pair (X, τ) consisting of a set X and topology τ on X.

Eg : ① $X = \{a, b, c\}$

$\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ is a topology on X.

(2) $\tau = \{X, \phi, \{a, b\}, \{b, c\}\}$

$\{(a, b) \cap (a, c)\} = \{b\} \notin \tau$ is not a topology on X.

Discrete topology :-

If X is any set, the collection of all subset of X is a topology on X such a topology is called Discrete topology.

Indiscrete topology (or) Trivial topology.

If X is any set, the collection consisting of X and ϕ only its called topology such topology is called as indiscrete topology (or) Trivial topology.

Finite complement topology :-

Let X be any set and $\mathcal{T}_f$ be the collection of all subset U of X such that $X-U$ is either finite or all of X then $\mathcal{T}_f$ is a topology on X such topology is called finite complement topology.

Theorem : 1

To show that union of U_α is in $\mathcal{T}_f$ if $\{U_\alpha\}$ is a family of non-empty elements of $\mathcal{T}_f$.

Proof:

Given $\{U_\alpha\}$ is a family of non-empty elements of $\mathcal{T}_f$.

$\Rightarrow X - U_\alpha$ is either finite or all X

To prove: $\cup U_\alpha$ is in $\mathcal{T}_f$.

To prove: $X - \cup U_\alpha$ is either finite or all of X.

$$X - \cup U_\alpha = \cap (X - U_\alpha)$$

Since $X - U_\alpha$ is finite

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$\Rightarrow \cap (X - U_\alpha)$ is also finite

$\Rightarrow X - \cup U_\alpha$ is finite

$\Rightarrow \cup U_\alpha \in \mathcal{J}_f$.

Hence $\cup U_\alpha$ is in $\mathcal{J}_f$.

2 If $U_1, U_2, \ldots U_n$ are non-empty elements of $\mathcal{J}_f$. then $\cup U_i$ is in $\mathcal{J}_f$.

Given $U_1, U_2, \ldots U_n$ are non-empty elements of $\mathcal{J}_f$.

$\Rightarrow X - U_i$ is finite of all of X.

To prove: $\cup U_i$ is in $\mathcal{J}_f$.

To prove; $X - \bigcup_{i=1}^{n} U_i$ is either finite or all of X.

$$X - \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} (X - U_i)$$

Since $X - U_i$ is finite.

$\Rightarrow \bigcap_{i=1}^{n} (X - U_i)$ is also finite

$\Rightarrow X - \bigcup_{i=1}^{n} U_i$ is finite

$\Rightarrow \bigcup_{i=1}^{n} U_i \in \mathcal{J}_f$

Hence $\bigcup_{i=1}^{n} U_i$ is in $\mathcal{J}_f$.

DEF **Basis for a topology :-**

Let X be any set and $\mathcal{B}$ is the non-empty collection of subsets, then $\mathcal{B}$ is said to be a basis for a Topology on X, if it satisfies the following two properties.

(i) For each $x \in X$. $\exists$ a basis element $B \in \mathcal{B}$, such that $x \in B$

(ii) If $x \in B_1 \cap B_2$ $\exists$ a basis element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_2 \cap B_2$

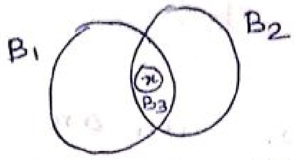
Topology $\mathcal{T}_f$ generated by $\mathcal{B}$

A Subset U of X is Said to be open in X [be the element of $\mathcal{T}$]

if for each $x \in U$ $\exists$ $B \in \mathcal{B}$

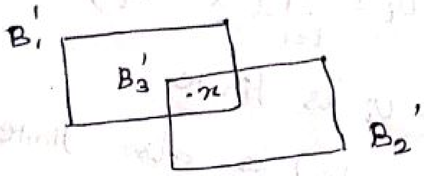
Such that $x \in B \subset U$.

Eg:



(1) Let $\mathcal{B}$ be the collection all circular regions in a plane then $\mathcal{B}$ is a basis

(2) Let $\mathcal{B}'$ be the collection of all rectangular regions in a plane then $\mathcal{B}'$ is a basis



Lemma: 1

Let x be a Set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X, then $\mathcal{T}$ equals the collection of all union of elements of $\mathcal{B}$.

Proof: Given: Let X be a Set.

$\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on x

To prove: $\mathcal{T}$ equals the Collection of all union of elements of $\mathcal{B}$.

Here the Collection of elements of $\mathcal{B}$ is also elements of $\mathcal{T}$.

Since $\mathcal{T}$ is a topology

$\Rightarrow$ This union is in $\mathcal{T}$.

Let $U \in \mathcal{T}$.

$\Rightarrow$ For each $x \in U$. $\exists$ $B \in \mathcal{B}$, s.t $x \in B \subset U$.

$$\Rightarrow U = \bigcup_{x \in U} B_x.$$

Hence U equals a union of elements of $\mathcal{B}$.

Lemma : 2

Let X be a topological space, Suppose that $\mathcal{C}$ is a collection of open set of X such that for each open set U on X and each x in U there is an element c of $\mathcal{C}$, such that $x \in c \subset U$, then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on X.

proof:

Let X be a topological space

$\mathcal{C}$ is a collection of open set ~~of~~ of X.

Such that for each open sets U of X.

To prove: $\mathcal{C}$ is a basis for a topo. $\mathcal{T}$.

(i) For each $x \in X$, $\exists$ $c \in \mathcal{C}$ Such that $x \in C$.

(ii) If $x \in c_1 \cap c_2$, $\exists$ $c_3 \in \mathcal{C}$ Such that $x \in c_3 \subset c_1 \cap c_2$

(i) Let $x \in X$.

Since X itself is an open set

$\Rightarrow x \in X$ $\exists$ $c \in \mathcal{C}$ Such that $x \in c \subset X$.

$\therefore$ The condition is satisfied.

(ii) Let $x \in c_1 \cap c_2$ where $c_1, c_2 \in \mathcal{C}$.

Since $c_1, c_2 \in \mathcal{C}$.

$\Rightarrow c_1$ and c_2 are open set in X.

$\Rightarrow c_1 \cap c_2$ is open in X.

$\Rightarrow$ $\exists$ $c_3 \in \mathcal{C}$, $x \in c_3 \subset c_1 \cap c_2$.

$\therefore$ The Condition is Satisfied

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Theorem : 3

Let $\mathcal{J}$ be the collection of open sets of X and $\mathcal{J}'$ be the topology generated by $\mathcal{C}$. We have to show that $\mathcal{J} = \mathcal{J}'$ (or)

Let $\mathcal{J}$ be the collection of open set of X then topology $\mathcal{J}'$ generated by $\mathcal{C}$ equals the topology.

Proof: $\mathcal{J} = \mathcal{J}'$

(ie) To prove $\mathcal{J} \subset \mathcal{J}'$ and $\mathcal{J}' \subset \mathcal{J}$

First To prove that $\mathcal{J} \subset \mathcal{J}'$

At $u \in \mathcal{J} \Rightarrow u$ is open in X

[$\because$ by (1) defn generated by $\mathcal{C}$]

$\Rightarrow \forall\ x \in u,\ \exists\ c \in \mathcal{C}$.

$\Rightarrow u \in \mathcal{J}'$, [$\because \mathcal{J}'$ is generated by $\mathcal{C}$]

$\therefore \mathcal{J} \subset \mathcal{J}'$ — (2)

Next to prove $\mathcal{J}' \subset \mathcal{J}$.

Let $w \in \mathcal{J}'$, $\mathcal{C}$ is a basis for topology $\mathcal{J}'$.

$\Rightarrow w = \bigcup C_\alpha,\ C_\alpha \in \mathcal{C}$ (by lemma 1)

The elements of $\mathcal{C}$ is also an element of $\mathcal{J}$. Since $\mathcal{J}$ is topology

$\therefore$ Their union is in $\mathcal{J}$.

$\Rightarrow U_\alpha \in \mathcal{J} \Rightarrow w \in \mathcal{J}$.

$\therefore \mathcal{J}' \subset \mathcal{J}$ — (3)

From (2) & (3).

$\mathcal{J} = \mathcal{J}'$

Lemma: 3

Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the topologi $\mathcal{J}$ and $\mathcal{J}'$ respy on X, then the following are equivalent.

(1) $\mathcal{J}'$ is finer than $\mathcal{J}$.

(2) For each $x \in X$ and each $B \in \mathcal{B}$ containing x $\exists$ $B' \in \mathcal{B}'$ s.t $x \in B' \subset B$.

proof:

Given $\mathcal{B}$ and $\mathcal{B}'$ be the basis for topology $\mathcal{J}$ and $\mathcal{J}'$ respy.

To prove: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (i)

Assume that $\mathcal{J}'$ is finer than $\mathcal{J}$.

To prove: For each x in X and $B \in \mathcal{B}$ containing in x there exists $B' \in \mathcal{B}'$, $x \in B' \subset B$.

Let $x \in X$ and $B \in \mathcal{B}$.

Since $\mathcal{B}$ is a basis topology.

$\Rightarrow x \in B$

Since $B \in \mathcal{J}$ and $\mathcal{J}' \subset \mathcal{J}$

$\Rightarrow B \in \mathcal{J}'$

$B' \in \mathcal{B}'$

$x \in B' \subset B$

Hence proof.

To prove (ii) $\Rightarrow$ (i).

Assumed that for each $x \in X$ for each $B \in \mathcal{B}$ $x \in B$, $B' \in \mathcal{B}'$, $x \in B' \subset B$.

To prove: $\mathcal{J}'$ is finer than $\mathcal{J}$ (ie) (or) $\mathcal{J} \subset \mathcal{J}'$

Let $U \in \mathcal{J} \Rightarrow x \in U$, $B \in \mathcal{B}$, $x \in B \subset U$ — (1)

By our assumption,

$B' \in \mathcal{B}'$. $x \in B' \subset B$ — (2)

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From ① and ②,

$\Rightarrow x \in B' \subset B \subset U$

$\Rightarrow x \in B' \subset U$

$\Rightarrow U \in \mathcal{J}'$

$\therefore \mathcal{J} \subset \mathcal{J}'$

Hence the proof.

DEF : Standard topology

If $\mathcal{B}$ is the collection of all open intervals (a, b) in the real line.

$(a,b) = \{x \mid a < x < b\}$

(ie) $\mathcal{B} = \{(a, b) \mid a, b \in R, a < b\}$

The topology generated by $\mathcal{B}$ is called the Standard topology on the real line.

Lower limit topology :-

If $\mathcal{B}'$ is the collection of all half open intervals of the form $[a, b)$ in the real line.

$[a,b) = \{x \mid a \leq x < b\}$

(ie) $\mathcal{B}' = \{[a, b) \mid a, b \in R, a < b\}$

The topology generated by $\mathcal{B}'$ is called the lower limit topology, on the Real line R where R is the given the lower limit topology we denote it by R_e.

K - Topology :-

Let $K = \{1/n \mid n \in Z^+\}$ and

$\mathcal{B}$ be its collection of all open intervals (a, b) along with all sets of the form

$(a, b) - K$.

The topology generated by $\mathcal{B}''$ is called K topology on R.

when $\mathbb{R}$ is given this topology we denote it by $\mathbb{R}_K$.

Lemma: 4

The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$ but are not comparable with one another.

<u>proof:</u> Given $\mathbb{R}_\ell$ is the lower limit topology

$\mathbb{R}_K$ is the k-topology and

$\mathbb{R}$ is the standard topology.

Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ be the topologies of $\mathbb{R}, \mathbb{R}_\ell, \mathbb{R}_K$

(i) To prove the topology of $\mathbb{R}_\ell$ is strictly finer the topology on $\mathbb{R}$.

(ii) T.P $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

(ie) $\mathcal{T}' \supset \mathcal{T}$

(ie) T.P $\mathcal{T} \subset \mathcal{T}'$

Given a basis element $(a,b) \in \mathcal{T}$.

Take a point $x \in (a,b)$ —①.

Consider a basis element $[x, b) \in \mathcal{T}'$.

$\therefore \mathcal{T}' \supset \mathcal{T}$.

On the other hand

a basis element $[x, d) \in \mathcal{T}'$,

there is no open interval $(a,b) \in \mathcal{T}$ satisfying $x \in (a,b) \subset [x, d)$.

Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

(ii) To prove the topology of $\mathbb{R}_K$ is strictly finer than the topology of $\mathbb{R}$

(ie) T.P $\mathcal{J}''$ is Strictly finer than $\mathcal{J}$.

(ie) T.P $\mathcal{J}'' \supset \mathcal{J}$

(ie) T.P $\mathcal{J} \subset \mathcal{J}''$.

Given a basis element $(a,b) \in \mathcal{J}$.

Take a point $x \in (a,b)$

This same interval is a basis element for $\mathcal{J}''$ that contains x.

$\therefore \mathcal{J}'' \supset \mathcal{J}$.

On the Otherhand,

Given the basis element $B = (-1,1) - K \in \mathcal{J}''$

Take a point $0 \in B$

There is no open interval $(a,b) \in \mathcal{J}$ satisfying $0 \in (a,b) \subset B$.

Thus $\mathcal{J}''$ is Strictly finer than $\mathcal{J}$.

(iii) To prove: The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable with one another

(ie) T.P ; $\mathcal{J}' \not\supset \mathcal{J}''$ and $\mathcal{J}'' \not\supset \mathcal{J}'$

First T.P $\mathcal{J}' \not\supset \mathcal{J}''$.

(ie) T.P $\mathcal{J}'' \not\subset \mathcal{J}'$.

Given a basis element $B = (-1,1) - K \in \mathcal{J}''$.

Take a point $0 \in B$,

There is no interval $[a,b) \in \mathcal{J}'$ satisfying $0 \in [a,b) \subset B$.

$\therefore \mathcal{J}' \not\supset \mathcal{J}''$.

Second T.P $\mathcal{J}'' \not\supset \mathcal{J}'$

(ie) T.P $\mathcal{J}' \not\subset \mathcal{J}''$.

Given a basis element $[x,d) \in \mathcal{J}'$.

Take a point $x \in [x,d)$

There is no interval $B = (a, b) - k \in J''$

Satisfying $x \in B \subset [x, d)$.

$J'' \not\supset J'$

$\therefore$ The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable. Hence the proof.

Sub basis for a Topology:

A sub basis $\mathcal{S}$ for a topology J on x is the collection of subset of x whose union equal x. The topology generated by the subbasis $\mathcal{S}$ is defined to the collection J of all union of finite intersection of elements of $\mathcal{S}$.

From a set x, C is said to be a simple relation if the foll. results are holds.

(i) comparability $x \subset y$ or $y \subset x$

(ii) transitivity $x \subset y$, $y \subset z \Rightarrow x \subset z$.

(iii) non reflexive $x \not\subset x$

Order topology:

Let x be a set with a simple order relation. Let $\mathcal{B}$ be the collection of all sets of foll. types

① All open intervals $(a, b) \in X$

② All intervals of the form $[a_0, b)$ where a_0 is the smallest element of X.

③ All intervals of the form $(a, b_0]$ where b_0 is the largest element of X.

Then the collection $\mathcal{B}$ form a basis for a topology on X. Such a topology is called order topology

Note:-

If x has no smallest elements there are no sets of the type 2

If x has no largest element there are no sets of the type 3.

DEF :

If x is an ordered set and $a \in x$, there are four subsets of x that are called rays determined by a. They are foll.

$(a, \infty) = \{x \mid x > a\}$

$[-\infty, a) = \{x \mid x < a\}$

$[a, \infty) = \{x \mid x \geq a\}$

$(-\infty, a] = \{x \mid x \leq a\}$.

Sets of the 1^{st} two types are called open rays and the sets of the last two types are called closed rays.

The product topology on $X \times Y$:-

Let X and Y be a topological space $\mathcal{B}$ be the collection of all sets of form $U \times V$ where U is open in X and V is open Y.

(ie) $\mathcal{B} = \{U \times V \mid U$ is open in X and V is open in $Y\}$

Such a collection form a basis for a topology is called product topology on $X \times Y$.

Theorem : 5

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y. $\mathcal{D} = \{B \times C \mid B \in \mathcal{B}$ and $C \in \mathcal{C}\}$ then $\mathcal{D}$ is a basis for the topology of $X \times Y$.

Proof

Let X and Y be a topological space

Let $\mathcal{J}$ and $\mathcal{J}'$ be the topology on X and

Also given $\mathcal{B}$ is a basis for the topology on X

$\mathcal{C}$ is a basis " " on Y.

$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}$ and $C \in \mathcal{C}\}$

To prove: $\mathcal{D}$ is a basis for the topology of $X \times Y$. $\longrightarrow$ (a)

Let W be a open set in $X \times Y$

and take $x \times y \in W$.

Then T.P $\exists$ $B \times C \in \mathcal{D}$ such that $x \times y \in B \times C$

(a) $\Rightarrow \forall x \times y \in W \;\exists\; U \times V \in \mathcal{J} \ni$

$U \times V \subset W$ — ①

$x \times y \in U \times V \subset W$

$\Rightarrow \forall x \in U \;\exists\; B \in \mathcal{B} \ni x \in B \subset U$ — ②

$\mathcal{C}$ is a basis for the topology on Y

$\Rightarrow \forall y \in V \;\exists\; C \in \mathcal{C} \ni y \in C \subset V$ — ③

Sub (2), (3) in ① we get

$x \times y \in B \times C \subset U \times V \subset W$

$\Rightarrow x \times y \in B \times C \subset W$

$\Rightarrow \mathcal{D}$ is a basis for topology on $X \times Y$.

Projections : DEF

Let $\pi_1 : X \times Y \to X$ be defined by the equation $\pi_1(x, y) = x$.

Let $\pi_2 ; X \times Y \to Y$ be defined by the equation $\pi_2(x, y) = y$

The maps π_1 and π_2 are called the projections of $X \times Y$ onto its first and second factors

Hints : If U is open in X, then the set $\pi_1^{-1}(U) = U \times Y$ which is also open in $X \times Y$.

If V is open in Y, then $\pi_2^{-1}(V) = X \times V$ which is also open in $X \times Y$

Then intersection of these two sets is

$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$.

Theorem :

$S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$

is a subbasis for the product topology on $X \times Y$

Proof: Given $S = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ is open in } Y\}$

To prove S be the subbasis for the topology on $X \times Y$.

then to prove ~~subb~~ S is a subbasis for $\mathcal{J}$.

To prove : $\mathcal{J}$ is generated by S

It is enough to prove that $\mathcal{J} \subset \mathcal{J}'$.

Take an element $U \times V \in \mathcal{J}$.

where U is open in X and

V is open in Y.

$\Rightarrow U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$

which is finite intersection of elements S.

$\Rightarrow U \times V \in \mathcal{J}'$

$\Rightarrow \mathcal{J} \subset \mathcal{J}'$

$\mathcal{J}'$ be the topology generated by S.

DEF: Subspace topology

Let X be a topological Space with topology $\mathcal{J}$. If Y is a subset of X.

The collection $\mathcal{J}_Y = \{ Y \cap U \mid U \in \mathcal{J} \}$ is a topology on Y called subspace topology

Lemma: 5

If $\mathcal{B}$ is a basis for the topology of X, then the collection $\mathcal{B}_Y = \{ B \cap Y \mid B \in \mathcal{B} \}$ is a basis for the subspace topology on Y.

proof: Given $\mathcal{B}$ is a basis for a topology of X.

To prove: $\mathcal{B}_Y = \{ B \cap Y \mid B \in \mathcal{B} \}$ is a basis for the subspace topology on Y.

Let $U \cap Y$ is open in Y, where U is open in X.

Take $y \in U \cap Y$

Then to prove that $\exists B \cap Y \in \mathcal{B}_Y$ such that

$y \in B \cap Y \subset U \cap Y$

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Here $y \in U \cap Y$.

$\Rightarrow y \in U$ also U is Open in X

$\Rightarrow \forall y \in U$, $\exists$ $B \in \mathcal{B} \ni y \in \underline{B \subset U}$

$\Rightarrow y \in B \cap Y \subset U \cap Y$

$\therefore \mathcal{B}_Y$ is a basis for subspace topology on Y.

Lemma:

$B_1 \cap Y$

$B_2 \cap Y$

$B_3 \cap Y$

Let Y be a Subspace of X.

If U is an open in Y and Y is an open in X then U is Open in X.

Proof: Since U is open in Y, $U = Y \cap V$, for some set V open in X. Since Y and V are both open in X.

$\therefore Y \cap V$ is open in X (Intersection of open set is open)

Hence U is open in X.

Theorem: 6

If A is a subspace of X and B is a Subspace of Y, then the product topology on $A \times B$ is the Same as the topology $A \times B$ inherits as a Subspace of $X \times Y$.

Proof: Given, A is a Subspace of X and B is a Subspace of Y

To prove: The product topology on $A \times B$ is the Same as the topology $A \times B$ inherits as a Subspace of $X \times Y$.

To prove :

This set $U \times V$ is the general basis element of $X \times Y$ and U is open in X, V is open in Y.

Then $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$.

Since $U \cap A$ is a open set for subspace topology on A.

Also $V \cap B$ is a open set for subs. topology on B.

$(U \cap A) \times (V \cap B)$ is a basis element for product topology on $A \times B$.

Hence the basis for subspace topology on $A \times B$ and product topology on $A \times B$ are same

Closed set and limit point : DEF

A subset A of a topological space X is said to be closed if the set $X - A$ is open.

Eg: The subset $[a, b]$ of $\mathbb{R}$ is closed since $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, +\infty)$ is open.

Theorem: 7

Let X be a topological space, then following conditions hold.

(i) ϕ and X are closed.

(ii) Arbitrary intersections of closed sets are closed

(iii) finite unions of closed sets are closed.

Proof. Given : Let X be a topological space

To prove : (i) ϕ and X are closed

T.P $X - \phi$ is open

Since X and ϕ are itself open

$\Rightarrow X - \phi = X$ which is open

Then ϕ is closed

To prove / X is closed

To prove, $X - X$ is open

$X - X = \phi$ which is open

Then X is closed.

To prove / (ii) Arbitrary intersection of closed sets are closed

T.P $\bigcap_{\alpha \in J} A_\alpha$ is closed

T.P $X - \bigcap_{\alpha \in J} A_\alpha$ is open

Consider $X - \bigcap A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$

Since each A_α is closed

$\Rightarrow X - A_\alpha$ is open

$\Rightarrow \bigcup (X - A_\alpha)$ is open

$\Rightarrow X - A_\alpha$ is open

$\Rightarrow A_\alpha$ is closed

To prove (iii) Finite unions of closed sets are closed

Let $\{A_1, A_2, \ldots A_n\}$ be a finite number of closed sets

then prove that $\bigcup_{i=1}^{n} A_i$ is closed

T.P $X - \bigcup A_i$ is open

$\Rightarrow X - \bigcup A_i = \bigcap_{i=1}^{n} (X - A_i)$

Since each A_i is closed

$\Rightarrow X - A_i$ is open

$\Rightarrow \bigcap (X - A_i)$ is open

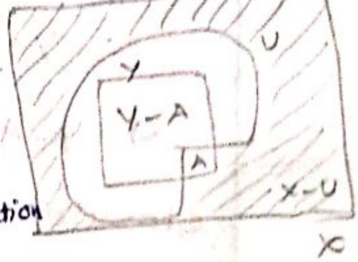
$\Rightarrow (X - \bigcup A_i)$ is open $\Rightarrow \bigcup A_i$ is closed

DEF.

If Y is a Subspace of X we Say that a Set A is closed in Y if A is a Subset of Y and if A is closed in the Subspace topology of Y.

Theorem: 8

Let Y be a Subspace of X. Then a Set A is closed in Y. iff equals the intersection of a closed set of X with Y.



proof: Given Y be a Subspace of X.

A is closed in Y.

To prove: A equals the intersection of the closed set of X with Y.

Part I: Since A is closed in Y

$\Rightarrow Y-A$ is open in Y and $A \subset Y$.

$\Rightarrow Y-A = U \cap Y$ where U is open in X.

By Fig, $X-U$ is closed in X.

So, it is enough to prove, $A = (X-U) \cap Y$.

Let $x \in A \Rightarrow x \notin Y-A$.

$\Rightarrow x \notin U \cap Y \Rightarrow x \in (X-U) \cap Y \therefore A = (X-U) \cap Y$

Part II: Given A equals the intersection of a closed set of X with $Y \longrightarrow$ (1)

T.P A is closed in Y.

(ie) T.P $Y-A$ is Open in Y.

(1) $\Rightarrow A = C \cap Y$, where C is closed in X

$\Rightarrow X-C$ is open in X.

Since Y is a subspace of X,

$\Rightarrow (X-C) \cap Y$ is open in Y.

So, it is enough to prove $Y - A = (X-C) \cap Y$

Let $x \in Y - A \Leftrightarrow x \in Y$ and $x \notin A$.

$\Leftrightarrow x \in Y \;,\; x \notin C \cap Y$

$\Leftrightarrow x \in Y \;,\; x \notin C$

$\Leftrightarrow x \in Y,\; x \in X - C$

$\Leftrightarrow x \in (X-C) \cap Y$

$\therefore Y - A = (X-C) \cap Y$ which is open in Y

$\Rightarrow Y - A$ is open in Y.

Closure and Interior of a set :-

Let X be a Topological space and A is the subset of X, then the interior of A is defined as the union of all open sets contained in A and it is denoted by Int A.

Let X be a topology space and A is a subset of X. Then the closure of A is defined as the intersection of all closed sets containing A and it is denoted by $cl(A)$ or $\bar{A}$.

Note :

(1) Interior of A is always an open set and $\bar{A}$ is always a closed set

(2) Int $A \subset A \subset \bar{A}$.

(3) Int A is the largest open set contained A and $\bar{A}$ is the smallest closed set containing A.

Theorem: 9.

Let Y be a Subspace of X. Let A be a Subsets of Y. Let $\bar{A}$ denote the closure of A in X, Then the closure of A in Y equals $\bar{A} \cap Y$.

Proof:

Let Y be a Subspace of X.

Let A be a Subset of Y.

Let $\bar{A}$ denoted the closure of A in X.

To prove: The closure of A in Y equals $\bar{A} \cap Y$,

Let B denote the closure of A in Y.

To prove: $B = \bar{A} \cap Y$

Since $\bar{A}$ is the closure of A in X.

$\Rightarrow \bar{A}$ is closed in X.

Since Y is a Subspace of X,

$\Rightarrow \bar{A} \cap Y$ is closed in Y.

Hence $\bar{A} \cap Y \supset A$.

Since B is the closure of A in Y.

$\Rightarrow B$ equals the intersection of all closed subset of Y containing A.

We must have $B \subset \bar{A} \cap Y \rightarrow (1)$

WKT B is closed in Y.

$\Rightarrow B = C \cap Y$, where C is closed in X.

Since $\bar{A}$ is the closure of A in X.

$\Rightarrow \bar{A}$ equals the intersection of all closed subset X containing A.

$\bar{A} \subset C$.

$\Rightarrow \bar{A} \cap Y \subset C \cap Y$

$\Rightarrow \bar{A} \cap Y \subset B \quad -(2)$

$(1),(2) \Rightarrow \bar{A} \cap Y = B$

$\therefore$ The closure of A in Y equals $\bar{A} \cap Y$.

Theorem: Let A be a subset of a topological space X.

(a) Then $x \in \bar{A}$ iff every open set U containing x intersects A

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ iff every basis element B containing x intersects A.

PROOF: (a) Consider the statement in (a)

It is the statement of the form $P \Leftrightarrow Q$

Let us transform each implication to its contrapositive, Then by obtaining the logically equivalent statement

$$\neg P \Leftrightarrow \neg Q \quad \text{(or)}$$

$$(\text{not } P) \Leftrightarrow (\text{not } Q)$$

ie To prove that $x \notin \bar{A}$ iff there exist open set U containing x does not intersect A

If $x \notin \bar{A}$ then $x \in X - \bar{A}$ and the set $U = X - \bar{A}$ is an open set containing x that does not intersect A.

Hence the I part.

Conversely if $\exists$ an open set U containing x that does not intersect A then $X - U$ is a closed set containing A.

Then by the definition of closure $\bar{A}$, $X - U$ is contained in $\bar{A}$.

$$\text{ie} \quad X - U \subset \bar{A}$$

$$\text{But} \quad x \notin X - U$$

$$\Rightarrow x \notin \bar{A}$$

Hence the proof of (a)

(b) To prove that $x \in \bar{A}$ iff every basis element B containing x intersects A

Let $x \in \bar{A}$

Let B be a basis element containing x

Since B is an open set by part (a), every basis element B containing x intersects A.

Conversely let us assume that every basis element B containing x intersects A.

To prove that $x \in \bar{A}$

If U is any open set such that $x \in U$, then by the definition of basis, $\exists$ a basis element B such that $x \in B \subset U$

Now by the hypothesis, every basis element B containing x intersects A

$\Rightarrow$ Every open set U containing x intersects A.

$\therefore$ By part (a) $\Rightarrow x \in \bar{A}$

Hence the proof.

<u>Note</u> (1) If U is an open set containing x then U is called a neighbourhood of x.

(2) If A is a subset of a topological space X then $x \in \bar{A}$ iff every neighbourhood of x intersects A.

<u>DEF:</u> <u>LIMIT POINT:</u> Let X be a topological space and $A \subset X$ and if $x \in X$ then we say that x is a limit point (or) cluster point (or) point of accumulation of A if every neighbourhood of x intersects A in some point other than x itself.

Theorem: Let A be a subset of a topological space X. Let A' be the set of all limitpoints of A. Then $\bar{A} = A \cup A'$.

Proof: Given that X is a topological space.

Let A be a subset of X

Also $A' = \{$ limit points of $A \}$

To prove that $\bar{A} = A \cup A'$

First let us prove that $\bar{A} \subset A \cup A'$

Let $x \in \bar{A}$

Then it is enough to prove that $x \in A \cup A'$

Case ① If $x \in A$ then $x \in A \cup A'$

Hence $\bar{A} \subset A \cup A'$

Case ② Let $x \notin A$

Since $x \in \bar{A}$, we have by theorem every open set U containing x intersects A

$\Rightarrow$ Every neighbourhood of x intersects A

Since $x \notin A$, the set U must intersect A in some point other than x

$\Rightarrow$ x is a limit point of A

$\Rightarrow$ $x \in A'$

$\therefore$ $x \in A \cup A'$

$\therefore$ $\bar{A} \subset A \cup A' \longrightarrow ①$

Next to prove that $A \cup A' \subset A$

Let $x \in A'$

$\Rightarrow$ x is a limit point of A

$\therefore$ By definition, every neighbourhood of x intersects A in some point other than x.

$\Rightarrow$ Every open set containing x intersects A

$\therefore$ By theorem $x \in \bar{A}$

$\therefore A' \subset \bar{A}$

But $A \subset \bar{A}$

$\Rightarrow A \cup A' \subset \bar{A} \longrightarrow$ (2)

$\therefore$ From (1) & (2) we prove that $\bar{A} = A \cup A'$

<u>Corollary:</u> A Subset of a topological space X is closed if and only if it contains all its limit points.

<u>PROOF:</u> Given that X is a topological space and A is a subset of X.

To prove that A is closed iff it contains all its limit points.

Let $A' = \{\text{Limit points of } A\}$

Then to prove that A is closed iff A contains A'

i.e. to prove A is closed iff $A' \subset A$

We know that A is closed $\Leftrightarrow A = \bar{A}$

$\Leftrightarrow A = A \cup A'$

$\Leftrightarrow A' \subset A$

$\therefore$ A is closed iff it contains all its limit points.

<u>HAUSDORFF SPACE</u>

A topological space X is called a Hausdorff space if for each pair of distinct points $x, y \in X$, there exists neighbourhoods U of x and V of y such that $U \cap V = \phi$

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Theorem: Every finite point set in a Hausdorff space X is closed.

Proof: Given that X is a Hausdorff space.

To prove that every finite point set in X is closed.

We know that every finite point set is a union of one point set.

So it is sufficient to show that every one point set $\{x_0\}$ is closed.

Let x be a point of X different from x_0

Since X is a Hausdorff space, there exists neighbourhoods U and V of x and x_0 that are disjoint (ie $U \cap V = \phi$)

Since U does not intersect $\{x_0\}$ then x cannot belong to the closure of the set $\{x_0\}$

$\therefore$ The closure of the set $\{x_0\}$ is $\{x_0\}$ itself so that it is closed.

$\therefore$ Every finite point set is closed.

T_1 - axiom: The condition that finite point sets be closed is called T_1 - axiom

DEF: Converges: Let X be a topological space and $\{x_n\}$ be a sequence points of X and $x \in X$. Then we say that the sequence $\{x_n\}$ converges to x if corresponding to each neighbourhood U of x there exists a positive integer N such that $x_n \in U$ for all $n \geq N$.

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Theorem: Let X be a topological space satisfying T_1-axiom. Let A be a subset of X. Then the point x is a limit point of A iff every neighbourhood of x contains infinitely many points of A.

Proof: Let X be a topological space satisfying T_1-axiom

$A \subset X$

To prove that x is a limit point of A iff every neighbourhood of x contains infinitely many points of A.

Part I: Assume that x is a limit point of A

To prove that every neighbourhood of x contains infinitely many points of A.

Suppose there exists a neighbourhood U of x intersects A in only finitely many points. $\rightarrow$ ①

$\Rightarrow$ U also intersects $A - \{x\}$ in finitely many points.

Let $U \cap (A - \{x\}) = \{x_1, x_2, x_3, \ldots x_m\}$

Since X satisfies T_1-axiom

$\Rightarrow$ Every finite point set is closed

$\Rightarrow \{x_1, x_2, x_3, \ldots x_m\}$ is closed.

$\Rightarrow X - \{x_1, x_2, x_3, \ldots x_m\}$ is open in X

$\therefore U \cap (X - \{x_1, x_2, x_3, \ldots, x_m\})$ is a open set containing x that does not intersect $A - \{x\}$

$\Rightarrow U \cap (X - \{x_1, x_2, x_3, \ldots, x_m\})$ is a neighbourhood of x that does not intersect $A - \{x\}$

This is a contradiction to ~~assumption~~ that x is a limit point of A

$\therefore$ our assumption ① is wrong.

$\therefore$ Every neighbourhood of x contains infinitely many points of A.

Part II Conversely let us assume that every neighbourhood of x contains infinitely many points of A $\longrightarrow$ ②

To prove that x is a limit point of A.

② $\Rightarrow$ Every neighbourhood of x intersects A in infinitely many points.

It certainly intersects A in some points other than x itself.

$\Rightarrow$ x is a limit point of A

Hence the proof.

Theorem: If X is a Hausdorff space then a sequence of points of X converges to atmost one point of X.

Proof: Given that X is a Hausdorff space.

Suppose that $\{x_n\}$ be a sequence of points of X

To prove that $\{x_n\}$ converges to $x \in X$

Let $y \neq x$

Since X is a Hausdorff space, there exists neighbourhoods U and V of x and y that are disjoint (ie $U \cap V = \phi$)

Since U contains x_n for all but finitely many values of n, the set V cannot.

$\Rightarrow$ The neighbouhood of y does not contain x_n

$\Rightarrow$ $\{x_n\}$ cannot converge to y

Hence the sequence $\{x_n\}$ of points of X converges to atmost one point of X.
