1 Elasticity

CHAPTER

1.1. INTRODUCTION

A body can be deformed (*i.e.*, changed in shape or size) by the suitable application of external forces on it. A body is said to be perfectly elastic, if it regains its original shape or size, when the applied forces are removed. This property of a body to regain its original state or condition on removal of the applied forces is called *elasticity*. A body which does not tend to regain its original shape or size, even when the applied forces are removed, is called a perfectly plastic body. No body, in nature, is either perfectly elastic or perfectly plastic. Quartz fibre is the nearest approach to a perfectly elastic body.

Stress : When an external force is applied on a body, there will be relative displacement of the particles and due to the property of elasticity, the particles tend to regain their original positions. *Stress is defined as the restoring force per unit area.* If a force F is applied normally to the area of cross-section A of a wire, then stress = F/A. Its dimensions are $ML^{-1}T^{-2}$.

Thermal Stresses : Suppose the ends of a rod are rigidly fixed, so as to prevent expansion or contraction. If the temperature of the rod is changed, tensile or compressive stresses, called thermal stresses, will be set up in the rod. If these stresses are very large, the rod may be stressed even beyond its breaking strength. The stress is *tensile* when there is an increase in length. The stress is *compressive* when there is a decrease in length. A tangential stress tries to slide each layer of the body over the layer immediately below it.

Strain : When a deforming force is applied, there is a change in length, shape or volume of the body. The ratio of the change in any dimension to its original value is called strain. It is of three types :-

(1) The ratio of change in length (l) to original length (L) is called *longitudinal strain* (l/L).

(2) Let $ABCD$ be a body with the side CD fixed (Fig. 1.1). Suppose a tangential force F is applied on the upper face AB. The shape of the body is changed to $A'B'CD$. The body is sheared by an angle ϕ. This angle ϕ measured in radians is called the *shearing strain* (ϕ).

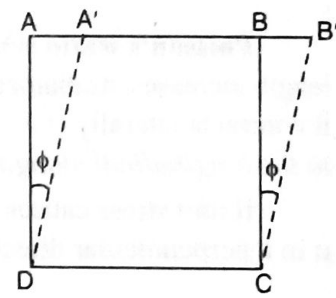

Fig. 1.1

(3) *Volume strain* (*Bulk strain*) : The ratio of change in volume (v) to original volume (V) is called *volume strain* (v/V).

Hooke's Law : *Within elastic limit, the stress is directly proportional to strain.* Stress $\propto$ strain or stress/strain = E. E is a constant called modulus of elasticity.

The dimensional formula of modulus of elasticity is $ML^{-1}T^{-2}$. Its units are Nm^{-2}.

1.2. DIFFERENT MODULI OF ELASTICITY

(1) **Young's modulus (E)** : *It is defined as the ratio of longitudinal stress to longitudinal strain within elastic limits.* Let a wire of length L and area of cross-section A undergo an increase in length l when a stretching force F is applied in the direction of its length.

Then, longitudinal stress = F/A and longitudinal strain = l/L.

$$\therefore \quad E = \frac{F/A}{l/L} = \frac{FL}{Al}$$

(2) **Rigidity modulus (G)** : *It is defined as the ratio of tangential stress to shearing strain.*

Consider a solid cube $ABCDEFGH$ (Fig. 1.2). The lower face $CDGH$ is fixed and a tangential force F is applied over the upper face $ABEF$. The result is that each horizontal layer of the cube is displaced, the displacement being proportional to its distance from the fixed plane. Point A is shifted to A', B to B', E to E' and F to F' through an angle ϕ, where $AA' = EE' = l$.

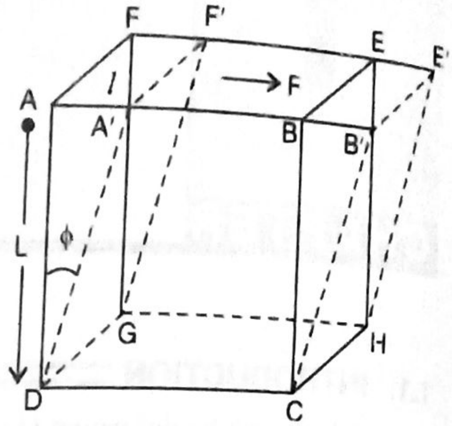

Fig. 1.2

Clearly $\phi = l/L$ where l is the relative displacement of the upper face of the cube with respect to the lower fixed face, distant L from it.

This angle ϕ through which a line originally perpendicular to the fixed face is turned, is a measure of the shearing strain.

$$\text{Now, Rigidity modulus } (G) = \frac{\text{Tangential stress}}{\text{Shearing strain}} = \frac{F/A}{\phi}$$

Here, $A = L^2$ = Area of face $ABEF$.

$\therefore$ $G = T/\phi$ where T = Tangential stress.

(3) **Bulk Modulus (K):** *It is defined as the ratio of volume stress (Bulk Stress) to the volume strain.*

When three equal stresses (F/A) act on a body in mutually perpendicular directions, such that there is a change of volume v in its original volume V, we have, Stress = pressure $P = F/A$. Volume strain = $-v/V$. The negative sign indicates that if pressure increases, volume decreases.

$$\therefore \quad K = \frac{\text{Bulk strees}}{\text{Volume strain}} = \frac{F/A}{-v/V} = \frac{P}{-v/V}$$

Poisson's Ratio (ν): When a wire is stretched, it becomes longer but thinner, *i.e.*, although its length increases, its diameter decreases. When a wire elongates freely in the direction of a tensile stress, it contracts laterally (*i.e.*, in a direction perpendicular to the force). *The ratio of lateral contraction to the longitudinal elongation is called Poisson's ratio.* It is denoted by the letter ν.

If unit stress causes an extensional strain λ in its own direction and lateral contractional strain μ in a perpendicular direction, $\nu = \mu/\lambda$.

1.3. RELATION BETWEEN ANGLE OF SHEAR AND LINEAR STRAIN

Consider a cube $ABCD$ having each side equal to L with its face DC fixed (Fig. 1.3). Let a shearing force acting along the face AB deform the cube into the rhomboid $A'B'CD$. The angle through which the face AD or BC has been turned is evidently the shearing strain. The diagonal DB has elongated to DB' while the other diagonal AC has shortened to $A'C$. Draw BK perpendicular to DB'. Since ϕ is very small, $\Delta BKB'$ may be assumed to be a right angled isosceles triangle and $\angle BB'K = 45°$. Further, $DB \approx DK$.

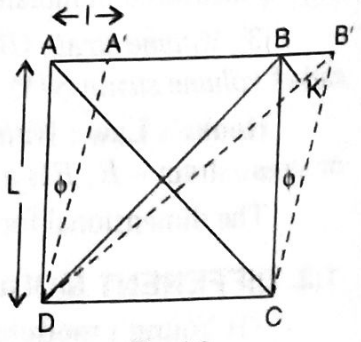

Fig. 1.3

$$\left.\begin{array}{r}\text{Tensile strain}\\ \text{along diagonal } DB\end{array}\right\} = e = \frac{DB' - DB}{DB} = \frac{DB' - DK}{DB} = \frac{KB'}{DB}$$

But in $\Delta BKB'$, $KB'/BB' = \cos 45°$

or $KB' = BB' \cos 45° = BB'/\sqrt{2}$

We have, $DB = \sqrt{L^2 + L^2} = L\sqrt{2}$

$$\therefore \quad e = \frac{KB'}{DB} = \frac{BB'/\sqrt{2}}{L\sqrt{2}} = \frac{BB'}{2L} = \frac{\phi}{2} \qquad [\because BB'/L = \phi]$$

Similarly, compressive strain along diagonal $AC = \phi/2$. Hence, a shearing strain ϕ is equivalent to a tensile strain and a compressive strain at right angles to each other, each of value $\phi/2$.

Similarly, it can be proved that a shearing stress is equivalent to a linear tensile stress and an equal compressive stress at right angles to each other.

1.4. RELATION BETWEEN VOLUME STRAIN AND LINEAR STRAIN

Consider a unit cube. Suppose it is subjected to three equal stresses, all tending to expand the cube, in three mutually perpendicular directions. Each side becomes $(1 + e)$ where e is the linear strain. Hence the new volume of the cube is $(1 + e)^3 \approx 1 + 3e$.

Increase in volume = $3e$.

$$\therefore \quad \text{Volume strain} = \frac{\text{increase in volume}}{\text{original volume}} = \frac{3e}{1} = 3e$$

1.5. WORK DONE IN A STRAIN

When a body is strained, work has to be done to deform the body. This work done is stored up in the body as potential energy. It can be shown that the work done per unit volume in any kind of strain (linear, shear or bulk) is equal to [$\frac{1}{2}$ × (stress) × (strain)].

***(a)* Linear Strain:** Let a force F act on a wire of length L and area of cross-section A such that the increase in length is l.

Young's modulus $= E = \frac{FL}{Al}$ or $F = \frac{EAl}{L}$.

Work done in producing a stretching $dl = F \, . \, dl = \frac{EAl}{L} dl$

$$\therefore \quad \left.\begin{array}{l}\text{Total work done to produce a}\\ \text{stretching of the wire from 0 to } l\end{array}\right\} = W = \int_0^l F.dl$$

$$= \int_0^l \frac{EAl}{L} dl = \frac{EA}{L}\left[\frac{l^2}{2}\right]_0^l = \frac{1}{2}\frac{EAl^2}{L} = \frac{1}{2}\cdot\frac{EAl}{L} l = \frac{1}{2}.F.l$$

$$= \frac{1}{2} \times \text{Stretching force} \times \text{Elongation produced.}$$

Now, volume of the wire = $A.L$.

Hence, work done per unit volume of the wire

$$= \frac{\frac{1}{2}F.l}{AL} = \frac{1}{2}\frac{F}{A}\cdot\frac{l}{L} = \frac{1}{2}\,\text{Stress} \times \text{Strain.}$$

(b) Shearing Strain: Let a tangential force F, acting in the direction AB of cube $ABCD$ shear it through an angle ϕ. In Fig. 1.3, $AD = L$ and $AA' = l$.

Stress = F/L^2 and shearing strain = $\phi = l/L$

$$\therefore \quad G = \frac{F/L^2}{l/L} = \frac{F}{Ll} \text{ or } F = GLl.$$

Work done to displace the layer AB by $dl = F\,dl = GLl\,dl$.

$$\therefore \left.\begin{array}{l}\text{Total work done during the}\\ \text{whole displacement from 0 to } l\end{array}\right\} = \int_0^l F\,dl = \int_0^l GLl\,dl$$

$$= \frac{1}{2} GL \cdot l^2 = \frac{1}{2} GLl \cdot l = \frac{1}{2} Fl$$

$$= \frac{1}{2} \text{ Tangential force} \times \text{displacement.}$$

Volume of the cube = L^3.

$$\text{Hence, work done per unit volume} = \frac{1}{2}\frac{Fl}{L^3} = \frac{1}{2}\frac{F}{L^2}\frac{l}{L}$$

$$= \frac{1}{2}(F/A).\phi$$

$$= \frac{1}{2} \text{ Stress} \times \text{Strain.}$$

(c) Volume Strain: Let a stress or pressure P be applied uniformly all over a body of volume V such that its volume decreases by v.

Stress = P and strain = v/V.

$$\text{Hence} \quad K = \frac{P}{v/V} \text{ or } P = \frac{Kv}{V}$$

Work done to produce a small decrease in volume $dv = P\,dv$

$$\therefore \left.\begin{array}{l}\text{Total work done for the whole}\\ \text{decrease in volume from 0 to } v\end{array}\right\} = \int_0^v P.\,dv$$

$$= \int_0^v \frac{Kv}{V}\,dv = \frac{1}{2}\frac{Kv^2}{V} = \frac{1}{2}\cdot\frac{Kv}{V}v = \frac{1}{2}P.v$$

$$= \frac{1}{2} \text{ stress} \times \text{change in volume.}$$

$$\text{Therefore, work done per unit volume} = \frac{1}{2}\frac{Pv}{V} = \frac{1}{2}P.\frac{v}{V}$$

$$= \frac{1}{2} \text{ Stress} \times \text{Strain}$$

Example 1: *Find the energy stored in a wire 5 metres long and 10^{-3} metre in diameter when it is stretched through 3×10^{-3} metre by a load. Young's modulus of material is 2×10^{11} Nm^{-2}.*

The energy stored in the wire is equal to the work done in stretching it.

$$W = \frac{1}{2} \text{ Stretching force} \times \text{Elongation produced.}$$

Since $E = \dfrac{F/A}{l/L}$, we have, $F = \dfrac{EAl}{L}$

Work done $= \dfrac{1}{2}\dfrac{EAl}{L}\, l = \dfrac{1}{2}\dfrac{EAl^2}{L}$

Here $E = 2 \times 10^{11}$ Nm^{-2}; $r = 0.5 \times 10^{-3}$ m

$A = \pi r^2 = \pi (0.5 \times 10^{-3})^2 = \pi \times 0.25 \times 10^{-6}$ m^2

$l = 3 \times 10^{-3}$ m; $L = 5$ m

$$W = \frac{1}{2}\frac{2 \times 10^{11} (\pi \times 0.25 \times 10^{-6})(3 \times 10^{-3})^2}{5} = 0.1414 \text{ J.}$$

Example 2: *Calculate the elastic energy stored up in a wire originally 5 metres long and 10^{-3} m in diameter which has been stretched by 3×10^{-4} m due to a load of 10 kg.*

$W = \dfrac{1}{2}$ Stretching force × Elongation produced $= \dfrac{1}{2} F \times l$

Stretching force $= F = 10 \times 9.8$ N $= 98$ N

Elongation produced $= l = 3 \times 10^{-4}$ m

$\therefore \quad W = \dfrac{1}{2} \times 98 \times 3 \times 10^{-4} = 1.47 \times 10^{-2}$ J.

Example 3: *A steel wire 2×10^{-3} m in diameter is just stretched between two points at a temperature of 20°C. Determine its tension when the temperature falls to 10°C. Linear expansivity of steel = 1.1×10^{-5} K^{-1}. Young's modulus for steel = 2.1×10^{11} Nm^{-2}.*

The total contraction of the wire when the temperature falls by 10° C $\Big\} = L\alpha t$

i.e., $l = L \times 1.1 \times 10^{-5} \times 10 = L \times 1.1 \times 10^{-4}$

$\therefore \quad F = \dfrac{EAl}{L} \qquad \left(\because E = \dfrac{F/A}{l/L}\right)$

$$= \frac{2.1 \times 10^{11} \times \pi (10^{-3})^2\, L \times 1.1 \times 10^{-4}}{L} = 72.56 \text{ N.}$$

1.6. BEHAVIOUR OF A WIRE UNDER PROGRESSIVE TENSION

If we subject a wire to gradually increasing load and plot a graph between load and extension, we obtain a curve of the form shown in Fig. 1.4. It is called the *stress–strain* diagram. In the part *OA*, which is straight, the extension is proportional to the load and the wire obeys Hooke's law. In this range, the wire regains its original length when unloaded and so it is called the range of *perfect elasticity*. At *A*, the wire reaches the elastic limit.

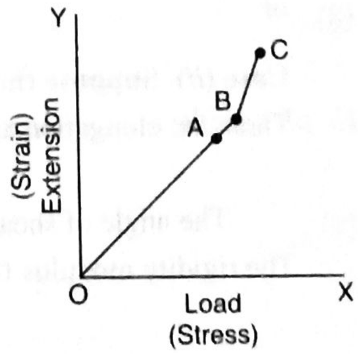

Fig. 1.4

If the wire is loaded beyond *OA*, the extension is no longer proportional to the load and Hooke's law is not obeyed. If the load is now removed, the wire will not regain its original length, but a permanent elongation will be produced in it. At the point *B*, even the addition of a very small load causes enormous elongation. This point is called the *yield point*. After the yield point, the extension increases very rapidly and depends on the time for which the load acts. The extension of the wire goes on increasing and the area of cross-section decreases until the breaking point is reached and finally the wire will break. The load at which the wire breaks is known as the *breaking weight*. The maximum load to which the wire can be subjected, divided by its original cross-sectional area, is called the *breaking stress*.

Elastic after-effect. It is found that even within elastic limit, practically all substances take some time to reach their original length after the deforming force is removed. This delay in recovering back the original condition on removal of deforming forces is called *elastic after-effect*. Quartz, phosphor bronze, silver and gold have very little elastic after-effect and hence they are used as suspension wires in Boys' experiment, quadrant electrometer, moving coil galvanometer, etc.

Elastic Fatigue. If a body is continuously subjected to stress and strain, it gets fatigued. Consider two torsional pendulums A and B having similar wires. A is set into vibration for a fairly long time continuously, while B is at rest. Now if A and B are set into vibration, with the same amplitude to start with, A comes to rest earlier than B. This is due to *elastic fatigue* of the suspension wire. Elastic fatigue can be removed by giving sufficient rest to the wire.

1.7. RELATION BETWEEN THE ELASTIC MODULI

Suppose three stresses P, Q and R are acting perpendicular to the three faces $ABCD$, $ADHE$ and $ABFE$ of a unit cube of an isotropic material (Fig. 1.5). Each one of these stresses will produce an extension in its own direction and a compression along the other two perpendicular directions. If λ is the extension per unit stress, the elongation along the direction of P will be λP. If μ is the contraction per unit length per unit stress, then the contraction along the direction of P due to the other two stresses will be μQ and μR.

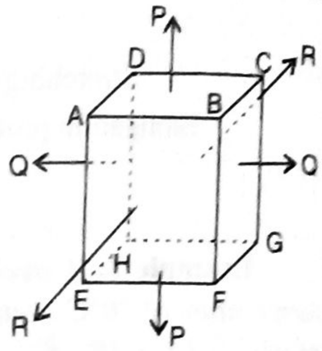

Fig. 1.5

Let all the three stresses act simultaneously on the cube.

Net elongation along the direction of $P = e = \lambda P - \mu Q - \mu R$.

Net elongation along the direction of $Q = f = \lambda Q - \mu P - \mu R$;

Net elongation along the direction of $R = g = \lambda R - \mu P - \mu Q$.

We can express the three elastic constants E, G and K in terms of λ and μ.

Case (*i*). Suppose only the stress P acts and $Q = R = 0$.

We have then the case of a simple longitudinal stress.

The linear strain $= e = \lambda.P$

$$\therefore \quad \text{The Young's modulus} = E = \frac{\text{Stress}}{\text{Linear strain}} = \frac{P}{\lambda.P} = \frac{1}{\lambda}$$

or
$$\lambda = \frac{1}{E} \qquad ...(1)$$

Case (*ii*). Suppose the stress $R = 0$ and $P = -Q$.

Then, the elongation along the direction of P is

$$e = \lambda.P - \mu(-P) = (\lambda + \mu)P$$

The angle of shear $\phi = 2e = 2(\lambda + \mu)P$

The rigidity modulus G is given by

$$G = \frac{\text{Stress}}{\text{Angle of shear}} = \frac{P}{\phi} = \frac{P}{2(\lambda + \mu)P} = \frac{1}{2(\lambda + \mu)}$$

or
$$2(\lambda + \mu) = \frac{1}{G} \qquad ...(2)$$

Case (*iii*). Let $P = Q = R$. Since the body is now subjected to uniform stress in all directions, the increase in volume is

$$3e = 3(\lambda - 2\mu)P \qquad [\because e = \lambda P - \mu P - \mu P = (\lambda - 2\mu)P]$$

$\therefore$ The bulk strain $= 3(\lambda - 2\mu) P$.

The bulk modulus $= K = \dfrac{\text{Stress}}{\text{Bulk strain}} = \dfrac{P}{3(\lambda - 2\mu)P} = \dfrac{1}{3(\lambda - 2\mu)}$

or $$(\lambda - 2\mu) = \frac{1}{3K} \quad \text{...(3)}$$

1. Relation between E, G and K

We have, from (2), $2(\lambda + \mu) = \dfrac{1}{G}$

or $$2\lambda + 2\mu = \frac{1}{G} \quad \text{...(4)}$$

From (3), $$\lambda - 2\mu = \frac{1}{3K} \quad \text{...(5)}$$

Adding (4) and (5), $$3\lambda = \frac{1}{G} + \frac{1}{3K} = \frac{3K + G}{3GK}$$

or $$\lambda = \frac{3K + G}{9GK} \quad \text{...(6)}$$

From (1), $$\lambda = 1/E$$

$\therefore$ $$E = \frac{9GK}{3K + G} \quad \text{...(7)}$$

2. Relation between G, K and ν

By definition, Poisson's ratio ν is given by $\nu = \mu/\lambda$.

From (2), $$\lambda + \mu = \frac{1}{2G}$$

From (3), $$\lambda - 2\mu = \frac{1}{3K}$$

Subtracting (3) from (2),

$$3\mu = \frac{1}{2G} - \frac{1}{3K}$$

$\therefore$ $$\mu = \frac{3K - 2G}{18GK} \quad \text{...(8)}$$

Hence $$\nu = \frac{\mu}{\lambda} = \frac{(3K - 2G)}{18GK} \times \frac{9GK}{(3K + G)} \quad \text{(Using Eq. 6)}$$

or $$\nu = \frac{(3K - 2G)}{6K + 2G} \quad \text{...(9)}$$

3. Relation between E, G and ν

We have, from (1), $\lambda = 1/E$

and from (2), $\lambda + \mu = 1/(2G)$

Dividing (2) by (1), $$\frac{\lambda + \mu}{\lambda} = \frac{E}{2G}$$

or $$1 + \frac{\mu}{\lambda} = \frac{E}{2G} \text{ or } 1 + \nu = \frac{E}{2G}$$

or $$\nu = \frac{E}{2G} - 1 \quad ...(10)$$

4. Limits to the value of ν.

We have from (9), $$\nu = \frac{3K - 2G}{6K + 2G}$$

or $$3K(1 - 2\nu) = 2G(1 + \nu) \quad ...(11)$$

Now both K and G must be positive quantities. Hence, if ν is positive, $(1 + \nu)$ is positive and R.H.S. is positive. Hence L.H.S. must be positive or $(1 - 2\nu) > 0$ or $\nu < \frac{1}{2}$. If ν is negative, $(1 - 2\nu)$ will be positive and hence $(1 + \nu)$ must be positive, or $\nu > -1$. Hence the theoretical limits to the value of ν are $\frac{1}{2}$ and -1. In actual practice, ν is always a positive quantity and lies between 0 and 0.5.

Example 4: *Calculate G and ν for silver, given E and K for silver = 7.25 × 10¹⁰ Nm⁻² and 11 × 10¹⁰ Nm⁻².*

Here, $$E = 7.25 \times 10^{10}\ Nm^{-2}\ ;\ K = 11 \times 10^{10}\ Nm^{-2}\ ;\ G = ?$$

Now, $$E = \frac{9GK}{3K + G}$$

$\therefore$ $$G = \frac{3KE}{9K - E} = \frac{3(11 \times 10^{10})(7.25 \times 10^{10})}{9(11 \times 10^{10}) - (7.25 \times 10^{10})}$$

$$= 2.607 \times 10^{10}\ Nm^{-2}.$$

Here, $$E = 7.25 \times 10^{10}\ Nm^{-2}\ ;\ G = 2.607 \times 10^{10}\ Nm^{-2}\ ;\ \nu = ?$$

$$\nu = \frac{E}{2G} - 1 = \frac{7.25 \times 10^{10}}{2 \times (2.607 \times 10^{10})} - 1 = 1.391 - 1 = 0.391.$$

1.8. DETERMINATION OF POISSON'S RATIO ν FOR RUBBER

A rubber tube R (like a cycle tyre) about 1 m in length and 2 cm in diameter is taken. It is suspended in a vertical position as shown in Fig. 1.6. The ends A and B are tightly closed with rubber corks. A graduated glass capillary tube (G) open at both ends is inserted inside the rubber tube through the upper end A. The rubber tube is completely filled with water till water rises in the tube G. A pointer P is fixed to the lower end B of the rubber tube. The lower end B carries a pan with weights (W).

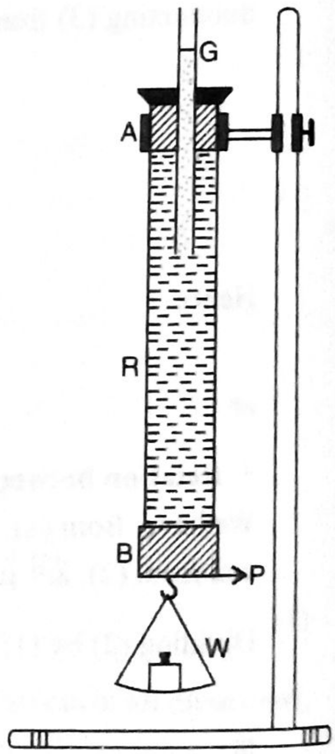

Fig. 1.6

When conditions become steady, the positions of the water meniscus in G and the pointer P are noted with the help of two separate travelling microscopes. When a suitable weight (W) is placed in the scale pan, the length of rubber tube increases and its area of cross-section decreases. The internal volume of the tube also increases. Consequently, the water level in G falls. The increase in the length of the tube is determined by noting the position of the pointer P in the microscope. The increase in volume is determined by noting the water level in the tube G. ν can be calculated by using the formula proved below.

Relation. Let L, D, A and V be the respective initial values of length, diameter, area of cross-section and internal volume of the rubber tube.

Then, $$V = A \times L.$$

When the load is applied, let dV be the increase in volume, dL the increase in length and dA the decrease in area of cross-section.

New volume

$$= V + dV = (A - dA)(L + dL)$$
$$= AL - L\,dA + A\,dL \qquad \text{(Neglecting the term } dA \times dL)$$
$$\therefore \quad dV = A\,dL - L\,dA \qquad ...(1)$$

Also we know that $A = \dfrac{\pi D^2}{4}$ or $dA = \dfrac{\pi D\,dD}{2}$

or
$$dA = \left(\frac{2A}{D}\right) dD \qquad ...(2)$$

Substituting this value of dA in Eq. (1)

$$dV = A\,dL - \left(\frac{2AL}{D}\right) dD \quad \text{or} \quad \frac{1}{A}\frac{dV}{dL} = 1 - \frac{2L}{D}\frac{dD}{dL}$$

or
$$\frac{1}{A}\frac{dV}{dL} = 1 - 2\frac{dD/D}{dL/L}$$

But,
$$\frac{dD/D}{dL/L} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \nu$$

$$\therefore \quad \frac{1}{A}\frac{dV}{dL} = 1 - 2\nu$$

or
$$\nu = \frac{1}{2}\left[1 - \frac{1}{A}\frac{dV}{dL}\right] \qquad ...(3)$$

TORSION

1.9. TORSION OF A BODY

When a body is fixed at one end and twisted about its axis by means of a torque at the other end, the body is said to be under torsion. Torsion involves shearing strain and so the modulus involved is the rigidity modulus.

Torsion of a cylinder–Expression for torque per unit Twist

Consider a cylindrical wire of length L and radius a fixed at its upper end and twisted through an angle θ by applying a torque at the lower end. Consider the cylinder to consist of an infinite number of hollow co-axial cylinders. Consider one such cylinder of radius x and thickness dx [Fig. 1.7(*i*)].

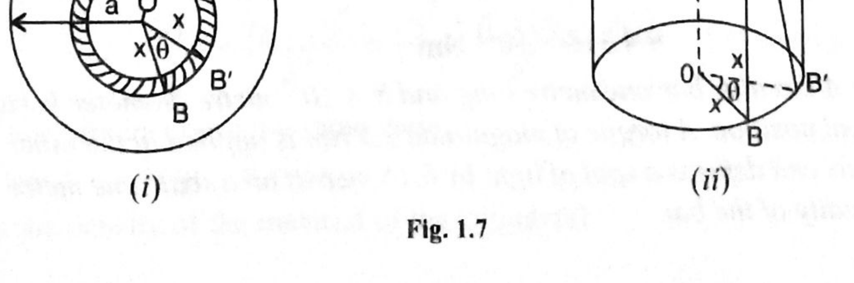

Fig. 1.7

A line such as AB initially parallel to the axis OO' of the cylinder is displaced to the position AB' through an angle ϕ due to the twisting torque [Fig. 1.7(*ii*)]. The result of twisting the cylinder is a shear strain. The angle of shear $= \angle BAB' = \phi$.

Now $\quad BB' = x.\theta = L\phi$ or $\phi = x.\theta/L$

We have, rigidity modulus $= G = \dfrac{\text{Shearing stress}}{\text{Angle of shear }(\phi)}$

$\therefore \quad$ shearing stress $= G\,.\,\phi = Gx\,\theta/L$

But, $\quad$ shearing stress $= \dfrac{\text{Shearing force}}{\text{Area on which the force acts}}$

$\therefore \quad$ Shearing force $=$ Shearing stress $\times$ Area on which the force acts.

The area over which the shearing force acts $= 2\pi\, x\, dx$

Hence, the shearing force $= F = \dfrac{G\,x\,\theta}{L} \times 2\pi x\, dx$

$$\left.\begin{array}{l}\text{The moment of this}\\ \text{force about the axis}\\ OO' \text{ of the cylinder}\end{array}\right\} = \frac{G\,x\,\theta}{L}\, 2\pi\, x\, dx . x = \frac{2\pi G\theta}{L} x^3\, dx$$

$$\therefore \quad \left.\begin{array}{l}\text{Twisting torque on}\\ \text{the whole cylinder}\end{array}\right\} = C = \int_0^a \frac{2\,\pi\, G\,\theta}{L} x^3\, dx$$

or
$$C = \frac{\pi\, G\, a^4\, \theta}{2L}$$

$$\left.\begin{array}{l}\text{The torque per unit twist (}i.e.,\\ \text{the torque when } \theta = 1 \text{ radian)}\end{array}\right\} = c = \frac{\pi\, G\, a^4}{2L}$$

Note 1: When an external torque is applied on the cylinder to twist it, at once an internal torque, due to elastic forces, comes into play. In the equilibrium position, these two torques will be equal and opposite.

Note 2: If the material is in the form of a hollow cylinder of internal radius a and external radius b, then,

$$\left.\begin{array}{l}\text{The torque acting}\\ \text{on the cylinder}\end{array}\right\} = C = \int_a^b \frac{2\,\pi\, G\,\theta}{L} x^3\, dx = \frac{\pi\, G\,\theta}{2L}(b^4 - a^4)$$

Torque per unit twist $= c = \pi G\,(b^4 - a^4)/(2L)$

Example 5: *What torque must be applied to a wire one metre long, 10^{-3} metre in diameter in order to twist one end of it through 90°, the other end remaining fixed ? The rigidity of the material of the wire is 2.8×10^{10} Nm^{-2}.*

Here, $L = 1$ m ; $G = 2.8 \times 10^{10}$ Nm^{-2} ; $a = \dfrac{10^{-3}}{2}$ m $= 0.5 \times 10^{-3}$ m ; $\theta = 90° = \pi/2$ radians ;

$$\therefore \quad C = \frac{\pi\, G\, a^4}{2L}\theta = \frac{\pi\,(2.8\times 10^{10})(0.5\times 10^{-3})^4}{2\times 1} \times \frac{\pi}{2}$$

$$= 4.318 \times 10^{-3} \text{ Nm}$$

Example 6: *A circular bar one metre long and 8×10^{-3} metre diameter is rigidly clamped at one end in a vertical position. A torque of magnitude 2.5 Nm is applied at the other end. As a result, a mirror fixed at this end deflects a spot of light by 0.15 metres on a scale one metre away. Calculate the modulus of rigidity of the bar.*

For a twist θ, the mirror turns through θ, and the reflected beam through 2θ. If the deflection is d on a scale D away, $2\theta D = d$ or $\theta = \frac{d}{2D} = \frac{0.15}{2 \times 1} = 0.075$ radians;

Here, $C = 2.5$ Nm ; $a = 4 \times 10^{-3}$ m ; $\theta = 0.075$ radians; $L = 1$ m ; $G = ?$

Hence, $C = \pi G a^4 \theta / 2L$ or $G = C \cdot 2L / \pi a^4 \theta$

i.e.,
$$G = \frac{2.5 \times 2 \times 1}{\pi (4 \times 10^{-3})^4 (0.075)} = 8.290 \times 10^{10} \text{ Nm}^{-2}$$

Example 7: *A steel wire of diameter 3.6×10^{-4} m and length 4 m extends by 1.8×10^{-3} m under a load of 1 kg and twists by 1.2 radians when subjected to a total torsional torque of 4×10^{-5} Nm at one end. Find the values of E, G and v for steel.*

We have, $E = \frac{F/A}{l/L} = \frac{FL}{A \cdot l}$

Here, $F = mg = 1 \times 9.8 = 9.8$ N : $L = 4$ m ;

$A = \pi a^2 = \pi(1.8 \times 10^{-4})^2$ m² and $l = 1.8 \times 10^{-3}$ m.

$$\therefore \quad E = \frac{9.8 \times 4}{\pi(1.8 \times 10^{-4})^2 \times 1.8 \times 10^{-3}} = 2.139 \times 10^{11} \text{ Nm}^{-2}$$

Torque which must be applied to twist one end of the wire through an angle θ radians $\Big\} = C = \frac{\pi G a^4 \theta}{2L}$

or $G = C\, 2L / \pi a^4 \theta$

Here, $C = 4 \times 10^{-5}$ Nm ; $L = 4$ m ; $a = 1.8 \times 10^{-4}$ m ;

$\theta = 1.2$ radians

$$\therefore \quad G = \frac{C\, 2\, L}{\pi\, a^4 \theta} = \frac{(4 \times 10^{-5}) \times 2 \times 4}{\pi\, (1.8 \times 10^{-4})^4 \times 1.2} = 0.8083 \times 10^{11} \text{ Nm}^{-2}$$

$$v = \frac{E}{2G} - 1 = \frac{2.139 \times 10^{11}}{2 \times 0.8083 \times 10^{11}} - 1 = 1.323 - 1 = 0.323.$$

Example 8: *Explain why a hollow rod is a better shaft than a solid one of the same mass, length and material.*

Consider a solid cylinder of length L, radius r and shear modulus G.

The torque required to twist the solid cylinder through an angle θ $\Big\} = C_1 = \frac{\pi G r^4 \theta}{2L}$ ...(1)

Let r_1 and r_2 be the inner and outer radii of the hollow cylinder of the same length, mass and material.

Then the torque required to twist it through the same angle θ is

$$C_2 = \frac{\pi G (r_2^4 - r_1^4)\theta}{2L} \quad ...(2)$$

Hence, $\frac{C_2}{C_1} = \frac{r_2^4 - r_1^4}{r^4} = \frac{(r_2^2 + r_1^2)(r_2^2 - r_1^2)}{r^4}$

Since the two cylinders have the same mass,

$$\pi (r_2^2 - r_1^2)\, l\rho = \pi r^2 l . \rho$$

(where ρ is the density of the material of the cylinders).

or $$r_2^2 - r_1^2 = r^2$$

Adding $2r_1^2$ to both sides, $$r_2^2 + r_1^2 = r^2 + 2r_1^2 \quad ...(3)$$

Hence, $$\frac{C_2}{C_1} = \frac{(r^2 + 2r_1^2)\,r^2}{r^4} = \frac{r^2 + 2r_1^2}{r^2} \quad ...(4)$$

or $$\frac{C_2}{C_1} = 1 + \frac{2r_1^2}{r^2}$$

$\therefore$ $C_2 > C_1$ *i.e.*, the twisting torque for a hollow cylinder is greater than that for a solid cylinder of the same mass, length and material. Hence a hollow cylinder is stronger and a better shaft than a solid one of the same mass, length and material.

1.10. DETERMINATION OF RIGIDITY MODULUS—STATIC TORSION METHOD

Searle's apparatus : The experimental rod is rigidly fixed at one end *A* and fitted into the axle of a wheel *W* at the other end *B* (Fig. 1.8). The wheel is provided with a grooved edge over which passes a tape. The tape carries a weight hanger at its free end. The rod can be twisted by adding weights to the hanger. The angle of twist can be measured by means of two pointers fixed at *Q* and *R* which move over circular scales S_1 and S_2. The scales are marked in degrees with centre zero.

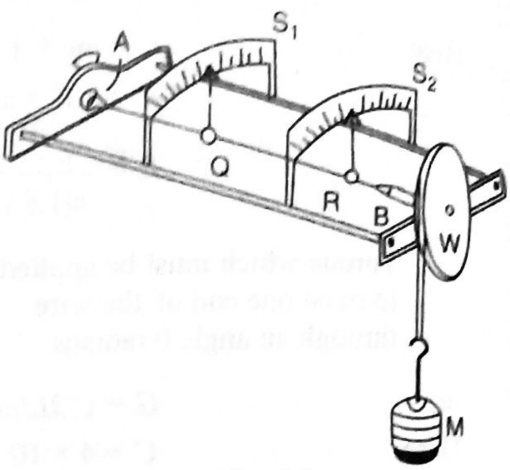

Fig. 1.8

With no weights on the hanger, the initial readings of the pointers on the scales are adjusted to be zero. Loads are added in steps of *m* kg (conveniently 0.2 kg). The readings on the two scales are noted for every load, both while loading and unloading. The experiment is repeated after reversing the twisting torque by winding the tape over the wheel in the opposite way. The observations are tabulated.

The readings in the last column give the twist for a load of *M* kg for the length *QR* (= *L*) of the rod.

The radius *a* of the rod and the radius *R* of the wheel are measured.

If a load of *M* kg is suspended from the free end of the tape, the twisting torque = *MgR*.

The angle of twist = θ degrees = θ. π/180 radians.

$$\therefore \quad \text{The restoring torque} = \frac{\pi G a^4}{2L} \cdot \frac{\theta \pi}{180}$$

For equilibrium, $$MgR = \frac{\pi G a^4}{2L} \cdot \frac{\theta . \pi}{180} \quad \text{or} \quad G = \frac{360\, M g R L}{\pi^2 a^4 \theta}$$

Since *a* occurs in the fourth power in the relation used, it should be measured very accurately.

Notes: (1) We eliminate the error due to the eccentricity of the wheel by applying the torque in both clockwise and anticlockwise directions.

(2) We eliminate errors due to any slipping at the clamped end by observing readings at two points on the rod.

1.11. DETERMINATION OF RIGIDITY MODULUS—STATIC TORSION METHOD. (SEARLE'S APPARATUS—SCALE AND TELESCOPE)

A plane mirror strip is fixed to the rod at a distance *L* from the fixed end of the rod [Fig. 1.9]. A vertical scale (*S*) and telescope (*T*) are arranged in front of the mirror. The telescope is focussed

Load	Reading on S_1				Mean θ_1	Reading on S_2				Mean θ_2	$\theta = \theta_2 - \theta_1$	θ for M kg
	Torque Clockwise		Torque Anticlockwise			Torque Clockwise		Torque Anticlockwise				
	Load increasing	Load decreasing	Load increasing	Load decreasing		Load increasing	Load decreasing	Load increasing	Load decreasing			

on the mirror and adjusted so that the reflected image of the scale in the mirror is seen through the telescope. With some dead load W on the weight-hanger, the reading of the scale division coinciding with the horizontal cross-wire is taken. Weights are added in steps of m kg and the corresponding scale readings are taken. Weights are then decreased continuously in steps of m kg and the readings taken again. The torque is reversed now, by passing the tape anticlockwise on the wheel. The readings are taken as before. From these readings, the shift in scale reading (s) for a load m kg is found.

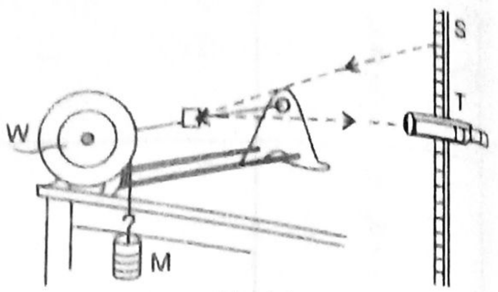

Fig. 1.9

The length L of the rod from the fixed end to the mirror is measured. The mean radius a of the rod is accurately measured with a screw gauge. The radius (R) of the wheel is found by measuring its circumference with a thread. The distance (D) between the scale and the mirror is measured with a metre scale.

G is calculated using the formula $G = \dfrac{4\,mg\,R\,L\,D}{\pi\,a^4 s}$

Load in kg	Telescope Reading						$\frac{X \sim Y}{2}$	Shift in scale reading for 4m kg
	Torque clockwise			Torque anticlockwise				
	Loading	Unloading	Mean (X)	Loading	Unloading	Mean (Y)		
W								
W + m								
W+ 2 m								
W + 3 m								
W + 4 m								
W + 5 m								
W + 6 m								
W + 7 m								

1.12. WORK DONE IN TWISTING A WIRE

Consider a cylindrical wire of length L and radius a fixed at its upper end and twisted through an angle θ by applying a torque at the lower end.

If c is the torque per unit angular twist of the wire, then the torque required to produce a twist θ in the wire is

$$C = c\,\theta.$$

The work done in twisting the wire through a small angle $d\theta$ is

$$C\,d\theta = c\theta\,d\theta.$$

$\therefore$ The total work done in twisting the wire through an angle θ $\Big\} = W = \int_0^\theta c.\theta\,d\theta$

or
$$W = \frac{1}{2}c\,.\,\theta^2$$

The work done in twisting the wire is stored up in the wire as potential energy.

Example 9: *Find the amount of work done in twisting a steel wire of radius 10^{-3} m and of length 0.25 m through an angle of 45°. Given G for steel = 8×10^{10} Nm^{-2}.*

We have,
$$W = \frac{1}{2}c.\theta^2 = \frac{1}{2}\frac{\pi G a^4}{2L}\theta^2 \qquad \left(\because c = \frac{\pi G a^4}{2L}\right)$$

Here, $G = 8 \times 10^{10}\ Nm^{-2}$; $a = 10^{-3}$ m ; $\theta = 45° = \pi/4$ rad ;

$L = 0.25$ m

$$\therefore \quad W = \frac{1}{2} \frac{\pi\,(8 \times 10^{10})(10^{-3})^4\,(\pi/4)^2}{2 \times 0.25} = 0.1550 \text{ J.}$$

1.13. TORSIONAL OSCILLATIONS OF A BODY

Suppose a wire is clamped vertically at one end and the other end carries a body (*i.e.*, a disc, bar or a cylinder) of moment of inertia I about the wire as the axis. Let the length, radius and rigidity modulus of the wire be respectively l, a and G. When the body is given a slight rotation by applying a torque, say by the hand, the wire is twisted. If the body is released, the body oscillates in the horizontal plane about the wire as axis. These oscillations are called *Torsional oscillations* and the arrangement is known as a *Torsion pendulum*.

Let us consider the energy of the vibrating system when the angle of twist is θ. Let ω be the angular velocity of the body.

The potential energy of the wire due to the twist $= \frac{1}{2} c.\theta^2$.

$$\left.\begin{array}{l}\text{The kinetic energy of the}\\ \text{body due to its rotation}\end{array}\right\} = \frac{1}{2} I\omega^2 = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2$$

$$\left.\begin{array}{l}\text{The total energy}\\ \text{of the system}\end{array}\right\} = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2} c\theta^2 = \text{constant}$$

Differentiating this with respect to t,

$$\frac{1}{2} I \,.\, 2 \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} + \frac{1}{2} c\, 2\theta \frac{d\theta}{dt} = 0.$$

or

$$I \frac{d^2\theta}{dt^2} + c\theta = 0 \text{ or } \frac{d^2\theta}{dt^2} + \frac{c}{I}\theta = 0$$

The body has simple harmonic motion and its period is given by

$$T = 2\pi \sqrt{\frac{I}{c}}$$

Rigidity modulus by Torsion pendulum (Dynamic torsion method) :

The torsion pendulum consists of a wire with one end fixed in a split chuck and the other end to the centre of a circular disc as in Fig. 1.10.

Two equal symmetrical masses (each equal to m) are placed along a diameter of the disc at equal distances d_1 on either side of the centre of the disc. The disc is rotated through an angle and is then released. The system executes torsional oscillations about the axis of the wire. The period of oscillations T_1 is determined.

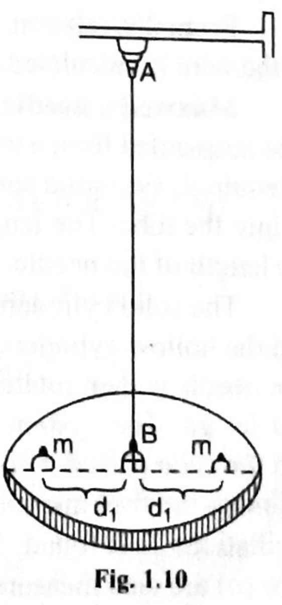

Fig. 1.10

Then

$$T_1 = 2\pi \sqrt{\frac{I_1}{c}}$$

or

$$T_1^2 = \frac{4\pi^2}{c} I_1.$$

Here, I_1 = Moment of inertia of the whole system about the axis of the wire and

c = torque per unit twist.

Let I_0 = M.I. of the disc alone about the axis of the wire.

i = M.I. of each mass about a parallel axis passing through its centre of gravity.

Then by the parallel axes theorem,

$$I_1 = I_0 + 2i + 2md_1^2$$

$$T_1^2 = \frac{4\pi^2}{c}\,[I_0 + 2i + 2m\,.\,d_1^2]. \quad ...(1)$$

The two masses are now kept at equal distances d_2 from the centre of the disc and the corresponding period T_2 is determined. Then,

$$T_2^2 = \frac{4\pi^2}{c}\,[I_0 + 2i + 2md_2^2] \quad ...(2)$$

$$\therefore \quad T_2^2 - T_1^2 = \frac{4\pi^2}{c}\cdot 2m\cdot(d_2^2 - d_1^2) \quad ...(3)$$

But $\quad c = \pi\, Ga^4/2L$

Hence $\quad T_2^2 - T_1^2 = \dfrac{4\pi^2\, 2m\,(d_2^2 - d_1^2)\, 2L}{\pi\, Ga^4}$

or $\quad G = \dfrac{16\,\pi\, Lm\,(d_2^2 - d_1^2)}{a^4\,(T_2^2 - T_1^2)}$

Using this relation, G is determined.

M.I. of the disc by torsional oscillations. The two equal masses are removed and the period T_0 is found when the disc alone is vibrating. Then,

$$T_0^2 = \frac{4\,\pi^2}{c}\, I_0 \quad \text{or} \quad I_0 = \frac{cT_0^2}{4\,\pi^2} \quad ...(4)$$

From (3), $\quad c = \dfrac{4\,\pi^2\, 2m\,(d_2^2 - d_1^2)}{T_2^2 - T_1^2}$

Hence $\quad I_0 = \dfrac{4\,\pi^2\, 2m\,(d_2^2 - d_1^2)}{T_2^2 - T_1^2}\cdot\dfrac{T_0^2}{4\,\pi^2} = \dfrac{2\,m\,(d_2^2 - d_1^2)\,T_0^2}{T_2^2 - T_1^2}$

From this relation, the moment of inertia of the disc about the axis of the wire is calculated.

Maxwell's needle: Maxwell's needle consists of a hollow metal tube suspended from a torsion head T by a wire whose rigidity modulus is required. Two solid and two hollow cylinders of equal lengths exactly fit into the tube. The length of each cylinder is equal to one quarter of the length of the needle.

The solid cylinders (S and S) are first placed in the inner position and the hollow cylinders (H and H) at the end as shown in Fig. 1.11(i). The needle is then rotated through a small angle about the wire as axis and let go. The system oscillates torsionally about the wire as axis and the time period T_1 is found. The positions of the solid and hollow cylinders are then interchanged [Fig. 1.11(ii)] and the period of torsional oscillations T_2 is found. The length of the wire (L) and the radius of the wire (a) are also measured.

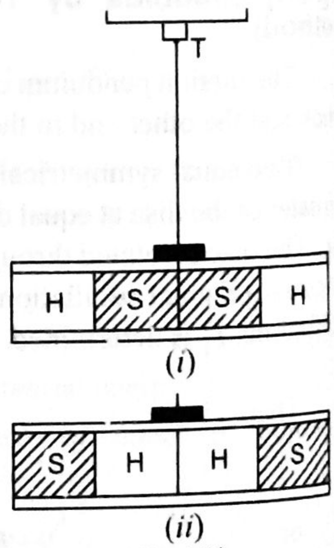

Fig. 1.11

Let I_0 = M.I. of the hollow tube about the wire as axis,

i_1 = M.I. of the solid cylinder about an axis through its C.G. parallel to the axis of the wire.

i_2 = M.I. of the hollow cylinder about an axis through its C.G. parallel to the axis of the wire,

m_1 = mass of the solid cylinder,

m_2 = mass of the hollow cylinder,

d = the length of each cylinder.

Then, the M.I. I_1 of the system about the axis of the wire in the first case is given by,

$$I_1 = I_0 + 2i_1 + 2i_2 + 2m_1 (d/2)^2 + 2m_2 (3d/2)^2 \quad ...(1)$$

Further, $T_1 = 2\pi\sqrt{I_1/c}$ or $T_1^2 = 4\pi^2 I_1/c$ ...(2)

Similarly, the M.I. I_2 of the system about the axis of the wire in the second case is given by,

$$I_2 = I_0 + 2i_1 + 2i_2 + 2m_2 (d/2)^2 + 2m_1 (3d/2)^2 \quad ...(3)$$

and $T_2^2 = 4\pi^2 I_2/c$ ...(4)

Subtracting (2) from (4), $T_2^2 - T_1^2 = \dfrac{4\pi^2}{c}(I_2 - I_1)$

Now, subtracting (1) from (3)

$$I_2 - I_1 = 4d^2 (m_1 - m_2) \text{ and } c = \pi\, Ga^4/2L$$

Hence, $$T_2^2 - T_1^2 = \frac{4\pi^2 \times 2L}{\pi G a^4} 4d^2 (m_1 - m_2)$$

or $$G = \frac{32\pi L d^2 (m_1 - m_2)}{(T_2^2 - T_1^2)\, a^4} \quad ...(5)$$

Example 10: *A metal disc of 0.1 m radius and mass 1 kg is suspended in a horizontal plane by a vertical wire attached to its centre. If the diameter of the wire is 10^{-3} m, its length 1 m, and the period of torsional vibrations is 4 seconds, find the rigidity modulus of the wire.*

The metal disc executes torsional oscillations about the axis of the wire. Hence

$$T = 2\pi\sqrt{I/c} \text{ or } T^2 = 4\pi^2 I/c,$$

where I = moment of inertia of the disc about the axis of the wire and

c = torque per unit twist = $\pi\, Ga^4/2L$.

$$\therefore \quad T^2 = 4\pi^2 \frac{I}{(\pi Ga^2/2L)} \text{ or } G = \frac{8\pi IL}{T^2 a^4}$$

Here, $I = MR^2/2$ [where M = mass of the disc = 1 kg and R = radius of the disc = 0.1 m].

$$\therefore \quad I = 1 \times (0.1)^2/2 = 0.005 \text{ kg m}^2.$$

$$L = \text{Length of the wire} = 1 \text{ m} ; a = 0.5 \times 10^{-3} \text{ m} ; T = 4 \text{ s}.$$

$$\therefore \quad G = \frac{8\pi IL}{T^2 a^4} = \frac{8\pi (0.005) \times 1}{4^2 (0.5 \times 10^{-3})^4} = 1.256 \times 10^{11} \text{ Nm}^{-2}$$

Example 11: *A steel bar is suspended in a horizontal position by a vertical wire attached to its centre. A horizontal torque of moment 5 Nm twists the bar horizontally through an angle of 12°. When the bar is released, it oscillates as a torsion pendulum with a period of $\frac{1}{2}$ s. Determine the moment of inertia.*

A torque of moment 5 Nm produces a twist of 12° or 12 × π/180 rad = 0.2090 rad.

Torque per unit twist = $c = 5/0.2090 = 23.88$ Nm.

Period of oscillation = $T = 2\pi\sqrt{I/c}$ or $T^2 = 4\pi^2 I/c$

∴ Moment of inertia = $I = cT^2/4\pi^2$.

Here, $c = 23.88$ Nm, $T = 0.5$ s

∴ $I = (23.88)(0.5)^2/4\pi^2 = 0.1513$ kg m².

Example 12: *A body, suspended symmetrically from the lower end of a wire, 1 m long, 1.22×10^{-3} m in diameter, oscillates about the wire as axis with a period of 1.25 seconds. If the modulus of rigidity of the material of the wire is 8×10^{10} Nm^{-2}, calculate the moment of inertia of the body about the axis of rotation.*

The body executes torsional oscillations about the axis of the wire with a time-period $T = 2\pi\sqrt{I/c}$

where I = M.I. of the body about the axis of the wire or the axis of rotation and

c = torque per unit twist = $\pi G \cdot a^4/2L$.

Hence $T^2 = 4\pi^2 I/c$.

or
$$I = T^2 \frac{c}{4\pi^2} = T^2 \frac{\pi G a^4}{2L\,4\pi^2}$$

∴
$$I = \frac{T^2 G a^4}{8\pi L}$$

Here, $T = 1.25$ s ; $G = 8 \times 10^{10}$ Nm^{-2} ; $a = 0.61 \times 10^{-3}$ m ; $L = 1$ m.

Hence
$$I = \frac{(1.25)^2(8\times10^{10})(0.61\times10^{-3})^4}{8\pi\times1}$$

$= 6.885 \times 10^{-4}$ kg m².

BENDING OF BEAMS

1.14. DEFINITIONS

Beam : *A beam is defined as a rod or bar of uniform cross-section (circular or rectangular) whose length is very much greater than its thickness.*

Bending Couple : If a beam is fixed at one end and loaded at the other end, it bends. The load acting vertically downwards at its free end and the reaction at the support acting vertically upwards constitute the bending couple. This couple tends to bend the beam clockwise. Since there is no rotation of the beam, the external bending couple must be balanced by another equal and opposite couple which comes into play inside the body due to the elastic nature of the body. The moment of this elastic couple is called the internal bending moment. When the beam is in equilibrium,

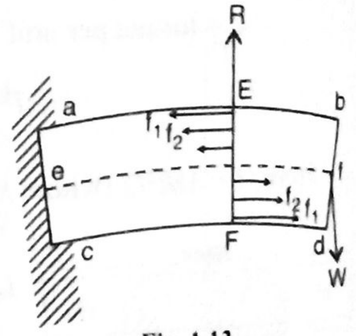

Fig. 1.12

the external bending moment = the internal bending moment.

Plane of Bending : The plane of bending is the plane in which the bending takes place and the bending couple acts in this plane. In Fig. 1.12, the plane of paper is the plane of bending.

Neutral Axis : When a beam is bent as in Fig. 1.12, filaments like *ab* in the upper part of the beam are elongated and filaments like *cd* in the lower part are compressed. Therefore, there must be a filament like *ef* in between, which is neither elongated nor compressed. Such a filament is known as the neutral filament and the axis of the beam lying on the neutral filament is the neutral axis. The change in length of any filament is proportional to the distance of the filament from the neutral axis.

1.15. EXPRESSION FOR THE BENDING MOMENT

Consider a portion of the beam to be bent into a circular arc, as shown in Fig. 1.13. *ef* is the neutral axis. Let R be the radius of curvature of the neutral axis and θ the angle subtended by it at its centre of curvature C.

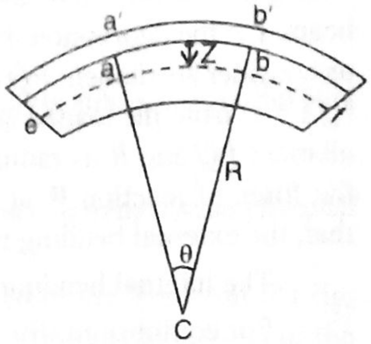

Fig. 1.13

Filaments above *ef* are elongated while filaments below *ef* are compressed. The filament *ef* remains unchanged in length.

Let $a'b'$ be a filament at a distance z from the neutral axis. The length of this filament $a'b'$ before bending is equal to that of the corresponding filament on the neutral axis ab.

We have, original length $= ab = R\theta$.

Its extended length $= a'b' = (R + z)\,\theta$

Increase in its length $= a'b' - ab = (R + z)\,\theta - R\theta = z\,.\,\theta$.

$$\therefore \quad \text{Linear strain} = \frac{\text{increase in length}}{\text{original length}} = \frac{z\,.\,\theta}{R\,.\,\theta} = \frac{z}{R}$$

If E is the Young's modulus of the material,

$$E = \text{Stress/Linear strain}$$

i.e., $$\text{Stress} = E \times \text{Linear strain} = E\,(z/R)$$

If δA is the area of cross-section of the filament,

the tensile force on the area δA = stress × area $= \dfrac{E\,.\,z}{R}\,\delta A$.

Moment of this force about the neutral axis *ef*

$$= \frac{E\,.\,z}{R}\,\delta A\,.\,z = \frac{E}{R}\,\delta A\,.\,z^2.$$

$$\left.\begin{array}{l}\text{The sum of the moments of}\\ \text{forces acting on all the filaments}\end{array}\right\} = \Sigma\,\frac{E}{R}\,\delta A\,.\,z^2$$

$$= \frac{E}{R}\,\Sigma\,\delta A\,.\,z^2$$

$\Sigma\delta A\,.\,z^2$ is called the geometrical moment of inertia of the cross-section of the beam about an axis through its centre perpendicular to the plane of bending. It is written as equal to Ak^2. *i.e.*, $\Sigma\delta A\,.\,z^2 = Ak^2$. ($A$ = Area of cross-section and k = radius of gyration).

But the sum of moments of forces acting on all the filaments is the internal bending moment which comes into play due to elasticity.

Thus, bending moment of a beam $= EAk^2/R$.

Notes : (*i*) For a rectangular beam of breadth b, and depth (thickness) d, $A = bd$ and $k^2 = d^2/12$.

$$\therefore \quad Ak^2 = bd^3/12.$$

(*ii*) For a beam of circular cross-section of radius r, $A = \pi r^2$ and $k^2 = r^2/4$.

$$\therefore \quad Ak^2 = \pi r^4/4.$$

(*iii*) EAk^2 is called the flexural rigidity of the beam.

1.16. DEPRESSION OF THE LOADED END OF A CANTILEVER

Cantilever : A cantilever is a beam fixed horizontally at one end and loaded at the other end.

Let OA be a cantilever of length l fixed at O and loaded with a weight W at the other end. OA' is the unstrained position of the beam. Let the depression $A'A$ of the free end be y (Fig. 1.14). Let us consider an element PQ of the beam of length dx at a distance ($QA = x$) from the loaded end. C is the centre of curvature of the element PQ and R its radius of curvature. The load W at A and the force of reaction W at Q constitute the external couple, so that, the external bending moment $= W.x$.

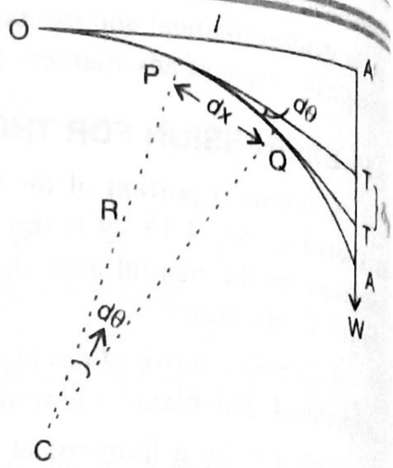

Fig. 1.14

The internal bending moment $= EAk^2/R$.

For equilibrium, $Wx = EA\,k^2/R$ or $R = EA\,k^2/Wx$ ...(1)

Draw tangents at P and Q meeting the vertical line at T and S respectively. Let $TS = dy$ and $d\theta$ = Angle between the tangents. Then, $\angle PCQ$ also $= d\theta$.

Now, $$PQ = dx = R\,d\theta \text{ or } d\theta = \frac{dx}{R} = dx \cdot \frac{Wx}{E\,A\,k^2}$$ (From Eq. 1)

We have, $$dy = x\,d\theta = x \cdot \frac{Wx\,dx}{EAk^2} = \frac{Wx^2\,dx}{E\,A\,k^2}$$ ...(2)

$\therefore$ the total depression of the end of the cantilever $$= y = \int_0^l \frac{Wx^2}{E\,A\,k^2}\,dx = \frac{W\,l^3}{3E\,A\,k^2}$$

Angle between the tangents at the ends of a cantilever :

Since the beam is fixed horizontally at O, the tangent at O is horizontal. If a tangent is drawn at A (the free end of the bent bar), it makes an angle θ with the horizontal.

Angle between the tangents at P and Q $$= d\theta = \frac{Wx}{EA\,k^2}\,dx.$$

Angle between the tangents at O and A $$= \theta = \int_0^l \frac{Wx}{E\,A\,k^2}\,dx$$

$$\therefore \quad \theta = \frac{W\,l^2}{2E\,A\,k^2}.$$

Work done in uniform bending. Consider a beam bent uniformly by an external couple. Let A be the area of cross-section of the beam. Consider a filament of area of cross-section δA at a distance z from the neutral axis (Fig. 1.13). Then,

the tensile force on the area $\delta A = \dfrac{Ez}{R}\,\delta A.$

The linear strain of this filament $= z/R$. If l is the length of the filament, then, the extension of the filament $= zl/R$.

The work done in bending the filament $$= \frac{1}{2}\text{ force} \times \text{extension}$$

$$= \frac{1}{2}\frac{Ez}{R}\,\delta A \times \frac{zl}{R} = \frac{1}{2}\frac{El}{R^2} \times z^2\,.\,\delta A.$$

For uniform bending R is constant. Hence, the work done in bending the whole beam is

$$W = \frac{1}{2}\frac{EI}{R^2}\Sigma z^2\,\delta A = \frac{1}{2}\frac{EI}{R^2} \times Ak^2 = \frac{1}{2}\frac{EAk^2}{R} \times \frac{l}{R}$$

Here, EAk^2/R = the bending moment and l/R = the angle subtended by the bent beam at its centre of curvature.

∴ The work done in uniform bending = $\frac{1}{2}$ (bending moment) × (Angle subtended by the bent beam at its centre of curvature).

Example 13 : *Obtain an expression for the depression at the free end of a heavy beam clamped horizontally at one end and loaded at the other end.*

Consider an element PQ of the beam of length dx at a distance x from the fixed end O (Fig. 1.14). Now, in addition to the load W acting at A, a weight equal to that of the portion $(l - x)$ of the beam also acts at its mid-point. Let W_1 be the weight of the beam. Then, the weight per unit length of the beam = W_1/l. Now, we have an additional weight $W_1(l - x)/l$ acting at a distance $(l - x)/2$ from Q. Therefore,

$$\left.\begin{array}{l}\text{total moment of the}\\ \text{external couple applied}\end{array}\right\} = W(l - x) + \frac{W_1}{l}(l - x)\frac{(l - x)}{2}$$

$$= W(l - x) + \frac{W_1}{2l}(l - x)^2$$

The beam being in equilibrium, this must be balanced by the bending moment EAk^2/R. Therefore,

$$W(l - x) + \frac{W_1}{2l}(l^2 - 2l\,.\,x + x^2) = \frac{EAk^2}{R} = EAk^2\left(\frac{d^2y}{dx^2}\right)$$

Integrating,

$$W\left(lx - \frac{x^2}{2}\right) + \frac{W_1}{2l}\left(l^2\,.\,x - l\,.\,x^2 + \frac{x^3}{3}\right) = EAk^2\frac{dy}{dx} + C$$

where C is a constant of integration.

Since at $x = 0$, $dy/dx = 0$, we have $C = 0$.

Integrating once again,

$$EAk^2\int_0^y dy = W\int_0^l (lx - x^2/2)\,dx + \frac{W_1}{2l}\int_0^l (l^2x - lx^2 + x^3/3)\,dx$$

or $$EAk^2y = W\left(\frac{l^3}{3}\right) + \frac{W_1}{2l}\left(\frac{l^4}{4}\right)$$

or $$EAk^2y = \frac{Wl^3}{3} + \frac{W_1l^3}{8}$$

or $$y = \left(W + \frac{3}{8}W_1\right)\frac{l^3}{3EAk^2}$$

1.17. MEASUREMENT OF E

(1) Cantilever depression : The given beam is clamped rigidly at one end (Fig. 1.15). A weight-hanger (H) is suspended at the free end of the beam. A pin (P) is fixed vertically by some wax at the free end of the beam. A travelling microscope (M) is focussed on the pin. The microscope is adjusted so that the horizontal cross-wire coincides with the tip of the pin and the reading on the vertical scale of the microscope is noted. Then weights m, 2 m, 3 m, 4 m, etc., are added to the weight-hanger. The microscope is adjusted each time to make the horizontal cross-wire coincide with the tip of the pin and the reading on the vertical scale of the microscope is noted in each case. Observations are made for decreasing loads also. The results are tabulated as follows:

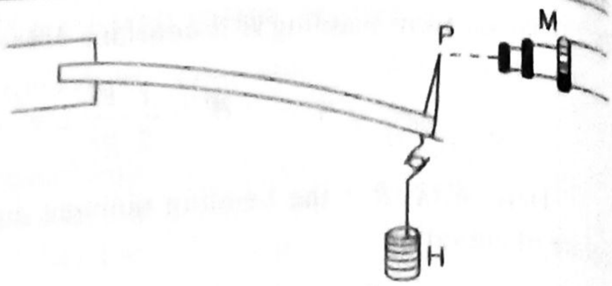

Fig. 1.15

Load in kg	Microscope Reading: Load increasing	Microscope Reading: Load decreasing	Mean	Depression for M (= 4 m)	Mean depression for a load of M kg
W			x		
$W + m$			x_1		
$W + 2m$			x_2		
$W + 3m$			x_3		
$W + 4m$			x_4	$x_4 - x$	
$W + 5m$			x_5	$x_5 - x_1$	
$W + 6m$			x_6	$x_6 - x_2$	
$W + 7m$			x_7	$x_7 - x_3$	

The mean depression (y) for a load M kg is found.

The length of the beam (l) between the clamped end and the loaded end is measured. The mean breadth (b) of the beam and its mean thickness (d) are determined.

If y is the depression produced for a load of Mg, then,

$$y = \frac{Mg\, l^3}{3EAk^2} \text{ or } E = \frac{Mg\, l^3}{3Ak^2 y}$$

Now, $$Ak^2 = bd^3/12 \text{ for a rectangular beam.}$$

Hence, $$E = \frac{Mg\, l^3}{3\,(b\,d^3/12)y} = \frac{4Mg\, l^3}{b\,d^3 y}.$$

The Young's modulus of the material of the beam is calculated using this relation.

(2) E – by measuring the tilt in a loaded cantilever.

The given rectangular beam is rigidly clamped at one end and a small plane mirror M is fixed at the free end [Fig. 1.16]. A weight-hanger (H) is attached at the free end of the beam. A vertical scale (S) and telescope (T) are arranged in front of the mirror. The telescope is focussed so that the image of the vertical scale due to reflection in the mirror is obtained in the telescope. The reading on the scale which coincides with the horizontal cross-wire is noted. Then weights m, 2 m, 3 m, 4 m etc., are added to the weight-hanger and the readings of the scale as observed in the telescope are noted in each case. Observations are made for decreasing loads also. The results are tabulated as follows :

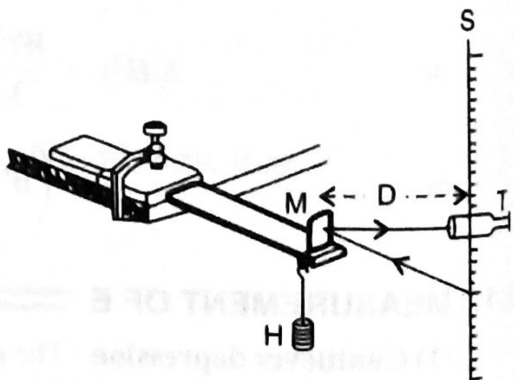

Fig. 1.16

Load in kg	Readings on the scale			Change in scale reading for M kg
	Load increasing	Load decreasing	Mean	

The average of the readings in the last column gives the mean change in scale reading s for a load of M kg. The distance D between the mirror and the scale is found. The breadth (b) and the thickness (d) of the beam are accurately measured.

The angle between the two ends of the centilever for a load of M kg $\Big\} = \theta = \dfrac{s}{2D}$ ...(1)

But $$\theta = \frac{Mg \cdot l^2}{2\,EA\,k^2} = \frac{Mg \cdot l^2}{2\,E \cdot bd^3/12} \qquad \left(\text{Since } A\,k^2 = \frac{b\,d^3}{12}\right)$$

$$\therefore \quad \theta = \frac{6\,Mgl^2}{bd^3\,E} \qquad ...(2)$$

From (1) and (2), $\dfrac{s}{2D} = \dfrac{6\,Mg \cdot l^2}{bd^3\,E}$ or $E = \dfrac{12\,Mg \cdot l^2 D}{bd^3 s}$

1.18. OSCILLATIONS OF A CANTILEVER

Let OA be a cantilever of length l, of negligible mass fixed at O. Let a mass M be attached at the other end A (Fig. 1.17). If the mass is slightly depressed and then released, the cantilever will execute simple harmonic motion about its original depressed position.

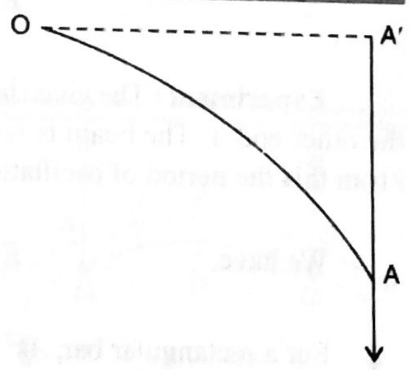

Fig. 1.17

The depression of the loaded end of the cantilever is

$$y = \frac{W\,l^3}{3\,E\,Ak^2}$$

or $$W = \frac{3\,E\,Ak^2}{l^3}\,y.$$

This must be equal to the elastic reaction of the cantilever balancing it and hence directed opposite to it.

If M is the mass of the weight W and d^2y/dt^2, the acceleration (upwards), we have,

$$\text{elastic reaction} = M\,\frac{d^2y}{dt^2}$$

$$\therefore \quad -M\,\frac{d^2y}{dt^2} = \frac{3\,E\,A\,k^2}{l^3}\,y$$

or $$\frac{d^2y}{dt^2} = \frac{-3\,E\,A\,k^2}{Ml^3}\,y$$

But, $$\frac{3\,E\,A\,k^2}{M\,l^3} = \text{A constant}$$

The acceleration of mass M or the free end of the cantilever is thus proportional to its displacement and is directed opposite to it.

It, therefore, executes a *S.H.M.* of time period T, given by

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{y}{\left(\frac{3EAk^2 y}{M.l^3}\right)}} = 2\pi\sqrt{\frac{Ml^3}{3EAk^2}}$$

If the mass of the cantilever is not negligible, it can be shown that,

$$T = 2\pi\sqrt{\frac{\left(M + \frac{1}{3}m\right)l^3}{3EAk^2}}$$ where m = mass of the cantilever.

The mass of the cantilever can be eliminated by finding the periods T_1 and T_2 for two different masses M_1 and M_2 attached to the cantilever at the same length. Then,

$$T_1^2 = 4\pi^2\frac{\left(M_1 + \frac{1}{3}m\right)l^3}{3EAk^2} \quad \text{and} \quad T_2^2 = 4\pi^2\frac{\left(M_2 + \frac{1}{3}m\right)l^3}{3EAk^2}$$

or

$$T_2^2 - T_1^2 = \frac{4\pi^2(M_2 - M_1)l^3}{3EAk^2}$$

$$\therefore \quad E = \frac{4\pi^2(M_2 - M_1)l^3}{3Ak^2(T_2^2 - T_1^2)}$$

Experiment : The given beam is rigidly clamped at O. A certain load of M_1 kg is suspended from the other end A. The beam is set in transverse oscillations and the time for 25 oscillations is found. From this the period of oscillation T_1 is calculated. Similarly, the period T_2 with a load M_2 is found.

We have,
$$E = \frac{4\pi^2(M_2 - M_1)l^3}{3Ak^2(T_2^2 - T_1^2)}$$

For a rectangular bar, $Ak^2 = bd^3/12$.

Hence,
$$E = \frac{16\pi^2 l^3 (M_2 - M_1)}{bd^3(T_2^2 - T_1^2)}$$

The length of the cantilever l, the breadth b and depth d are measured. E is calculated using the above formula.

Example 14 : *A steel bar, 0.3 m long, 2×10^{-2} m broad and 2×10^{-3} m thick is clamped at one end and loaded at the other with a mass of 0.01 kg. Calculate the period of vibration of the bar, neglecting the effect of weight of the bar. (E for steel = 20×10^{10} Nm^{-2}).*

For a cantilever of length l, loaded with a mass M, period of vibration $= T = 2\pi\sqrt{\frac{Ml^3}{3EAk^2}}$

Here, $l = 0.3$ m, $M = 0.01$ kg ; $E = 20 \times 10^{10}$ Nm^{-2}

$$Ak^2 = bd^3/12 = (2 \times 10^{-2})(2 \times 10^{-3})^3/12 = \frac{4}{3} \times 10^{-11}$$

$$\therefore \quad T = 2\pi \sqrt{\frac{(0.01)\times(0.3)^3}{3\times(20\times10^{10})\times\left(\frac{4}{3}\times10^{-11}\right)}} = 0.0365 \text{ second}$$

1.19. DEPRESSION AT THE MID-POINT OF A BEAM LOADED AT THE MIDDLE

Let AB represent a beam of length l, supported on two knife-edges at A and B and loaded with a weight W at the centre C. The reaction at each knife-edge is $W/2$ acting vertically upwards. The beam bends as shown in Fig. 1.18, the depression being maximum at the centre. The bending is non-uniform. Let $CD = y$.

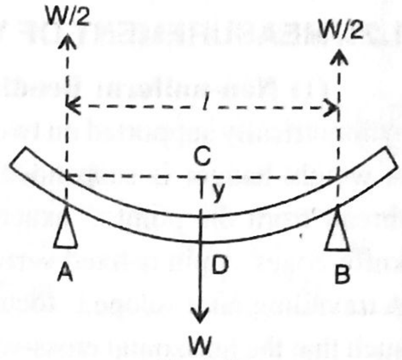

Fig 1.18

The portion DA of the beam may be considered as a cantilever of length $l/2$, fixed at D and bending upwards under a load $W/2$. Hence the elevation of A above D or,

the depression of D below $A = y = \dfrac{(W/2)\,(l/2)^3}{3EAk^2} = \dfrac{Wl^3}{48\,EAk^2}$

Note : The inclination of the tangent at the points A and B is given by

$$\tan\theta = \frac{dy}{dx} = \frac{W\,l^2}{16\,EAk^2}$$

Since θ is small, $\quad \tan\theta = \theta.$

$$\therefore \quad \theta = \frac{W\,l^2}{16\,EA\,k^2}$$

1.20. UNIFORM BENDING OF A BEAM

Consider a beam of negligible mass supported symmetrically on two knife-edges A and B in a horizontal level (Fig. 1.19). Let $AB = l$.

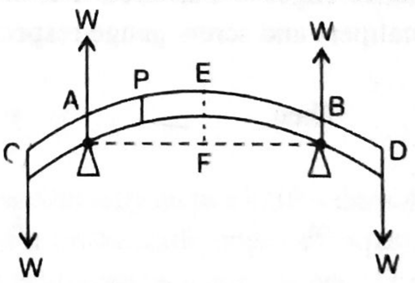

Fig. 1.19

Let equal weights W, W be added to the beam at its ends C and D. Let $AC = BD = a$. Then the beam is bent into an arc of a circle. The reactions on the knife-edges will then be W and W, acting vertically upwards. Consider the cross-section of the beam at any point P. The only forces acting on the part PC of the beam are the forces W at C and the reaction W at A.

The external bending moment with respect to P

$$= W\,.\,CP - W\,.\,AP = W(CP - AP) = W\,.\,AC = Wa.$$

This must be balanced by the internal bending moment EAk^2/R.

Hence, $\quad Wa = EAk^2/R \quad$ ...(1)

Since for a given load W, E, a and Ak^2 are constant, R is a constant. The bending is then said to be uniform. If y is the elevation of the mid-point of AB above its normal position (Fig. 1.20),

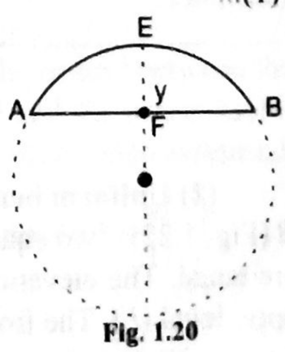

Fig. 1.20

$$EF\,(2R - EF) = AF^2$$

$$y\,(2R - y) = (l/2)^2$$

$$y\,.\,2R = l^2/4 \qquad (\because y^2 \text{ is negligible})$$

$$y = l^2/8R$$

From (1), $\frac{1}{R} = \frac{Wa}{E A k^2}$

$\therefore \quad y = \frac{Wal^2}{8 E A k^2}$

1.21. MEASUREMENT OF YOUNG'S MODULUS—BY BENDING OF A BEAM

(1) Non-uniform Bending : The given beam is symmetrically supported on two knife-edges (Fig. 1.21). A weight-hanger is suspended by means of a loop of thread from the point *C* exactly midway between the knife-edges. A pin is fixed vertically at *C* by some wax. A travelling microscope is focussed on the tip of the pin such that the horizontal cross-wire coincides with the tip of the pin. The reading in the vertical traverse scale of microscope is noted. Weights are added in equal steps of *m* kg and the corresponding readings are noted. Similarly, readings are noted while unloading. Th results are tabulated as follows :

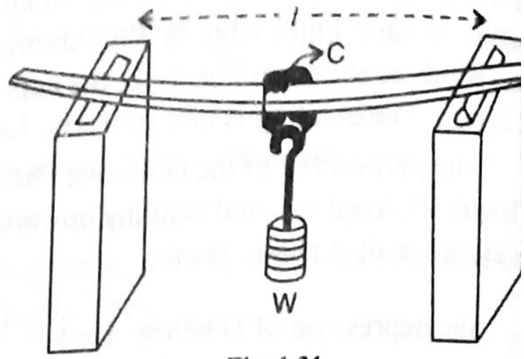

Fig. 1.21

Load in kg	*Readings of the microscope*			*y for M kg*
	Load increasing	*Load decreasing*	*Mean*	

The mean depression *y* is found for a load of *M* kg. The length of the beam (*l*) between the knife-edges is measured. The breadth *b* and the thickness *d* of the beam are measured with a vernier calipers and screw gauge respectively.

Then, $y = \frac{Wl^3}{48 EA k^2}$ or $E = \frac{Wl^3}{48 Ak^2 y}$

or $E = \frac{Mg\, l^3}{48 \times (bd^3/12) \times y}$ $\quad (\because W = Mg \text{ and } Ak^2 = bd^3/12)$

$\therefore \quad E = \frac{Mgl^3}{4\, bd^3 y}$

Example 15 : *In an experiment a rod of diameter 0.0126 m was supported on two knife-edges, placed 0.7 metre apart. On applying a load of 0.9 kg exactly midway between the knife-edges, the depression on the middle point was observed to be 0.00025 m. Calculate the Young's modulus of the substance.*

$$E = \frac{Mgl^3}{12 y \pi r^4} = \frac{(0.9)(9.8)\,(0.7)^3}{12\,(0.00025)\,\pi\,(0.0063)^4}$$

$\therefore \quad E = 2.039 \times 10^{11}\ \text{Nm}^{-2}$

(2) Uniform bending : The given beam is supported symmetrically on two knife-edges *A* and *B* (Fig. 1.22). Two equal weight-hangers are suspended, so that their distances from the kinfe-edges are equal. The elevations of the centre of the beam may be measured accurately by using a single optic level (*L*). The front leg of the single optic lever rests on the centre of the loaded beam and the hind legs are supported on a separate stand. A vertical scale (S) and telescope (T) are arranged in front of the mirror. The telescope is focussed on the mirror and adjusted so that the reflected image of the scale in the mirror is seen through the telescope. The load on each hanger is increased in equal steps of m kg and the corresponding readings on the scale are noted. Similarly, readings are noted while unloading. The results are tabulated as follows :

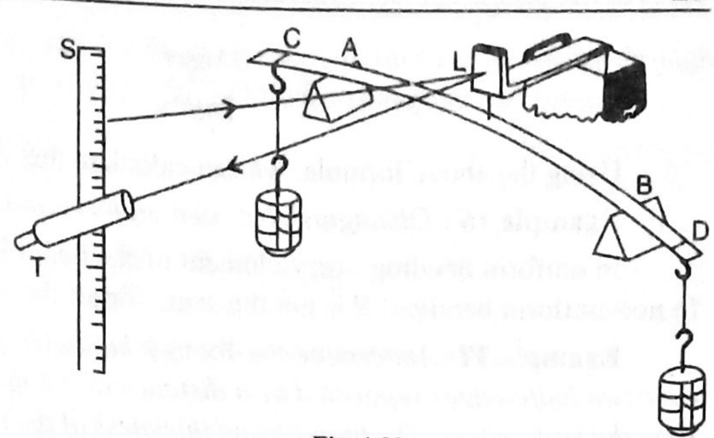

Fig. 1.22

Load in kg	*Readings of the scale as seen in the telescope*			*Shift in reading for M kg*
	Load increasing	*Load decreasing*	*Mean*	

The shift in scale reading for M kg is found from the table. Let it be S. If

D = The distance between the scale and the mirror,

x = the distance between the front leg and the plane containing the two hind legs of the optic lever,

then $$y = Sx/2D.$$

The length of the beam l between the knife-edges, and a, the distance between the point of suspension of the load and the nearer knife-edge ($AC = BD = a$) are measured. The breadth b and the thickness d of the beam are also measured.

Then, $$y = \frac{Wal^2}{8\,E\,A\,k^2} \quad \text{or} \quad \frac{Sx}{2D} = \frac{Mgal^2}{8\,E\,(b\,d^3/12)}$$

[Since $W = Mg$ and $Ak^2 = b\,d^3/12$]

$$\therefore \quad E = \frac{3\,Mgal^2 D}{Sxbd^3}$$

Pin and Microscope Method : The given beam is supported symmetrically on two knife-edges A and B. Two equal weight-hangers are suspended so that their distances from the knife-edges are equal. A pin is placed vertically at the centre of the beam. The tip of the pin is viewed by a microscope. The load on each hanger is increased in equal steps of m kg and the corresponding microscope readings are noted. Similarly, readings are noted while unloading. The results are tabulated as follows :

Load in kg	*Readings of the microscope*			*y for M kg*
	Load increasing	*Load decreasing*	*Mean*	

The mean elevation (y) of the centre for M kg is found. The length of the beam l between the knife-edges and a, the distance between the point of suspension of the load and the nearer knife-edge ($AC = BD = a$) are measured. The breadth b and the thickness d of the beam are also measured.

$$y = \frac{Wal^2}{8EAk^2} = \frac{Mgal^2}{8E(bd^3/12)} \quad \left(\because\ W = Mg \text{ and } Ak^2 = \frac{bd^3}{12}\right)$$

$$E = \frac{3Mgal^2}{2bd^3y}$$

Using the above formula, we can calculate the Young's modulus of the material of the beam.

Example 16 : *Distinguish between uniform and non-uniform bending.*

In uniform bending every element of the beam is bent with the same radius of curvature (R). In non-uniform bending, R is not the same for all the elements in the beam.

Example 17 : *Determine the Young's modulus of the material of a rod, if it is bent uniformly over two knife-edges separated by a distance of 0.6 m and loads of 2.5 kg are hung at 0.18 m away from the knife-edges. The breadth and thickness of the rod are 0.025 m and 0.005 m respectively. The elevation at the middle of the rod is 0.007 m.*

$$E = \frac{3Mgal^2}{2bd^3y} = \frac{3 \times 2.5 \times 9.8 \times 0.18 \times (0.6)^2}{2 \times 0.025 \times (0.005)^3 \times 0.007}$$

$$= 1.088 \times 10^{11}\ \text{Nm}^{-2}.$$

I Section of girders : A girder supported at its two ends as on the opposite walls of a room, bends under its own weight and/or, under the load placed above it. The middle portion gets depressed. The depression (y) at the mid-point of a rectangular beam is proportional to Wl^3/Ebd^3. For the depression (y) to be small for a given load (W), the length of the girder (l) should be small and its breadth (b), depth (d) and Young's modulus for its material (E) must be large.

Due to depression, the upper parts of the beam above the neutral surface contract, while those below the neutral surface extend. Hence the stresses have a maximum value at the top and bottom and progressively decrease to zero as we approach the neutral surface from either face. Therefore the upper and the lower surfaces of the beam must be stronger than the intervening part. That is why the two surfaces of a girder or iron rails (for railway tracks etc.) are made much broader than the rest of it, thus giving its cross-section the shape of the letter I. In this manner, material will be saved without appreciably impairing its strength.

1.22. SEARLE'S METHOD FOR DETERMINING E, G AND ν

To determine E : Two identical cylindrical bars AB and CD are suspended from a rigid support by two parallel torsionless strings attached to hooks at the mid-points of the bars. The centres of the bars are joined by the short wire of length l whose elastic constants are required. When the ends A and C of the bars are pulled closer to each other symmetrically, the wire is bent into a circular arc [Fig. 1.23(i)]. When the bars are released, the bars oscillate in the horizontal plane with supporting threads as axes.

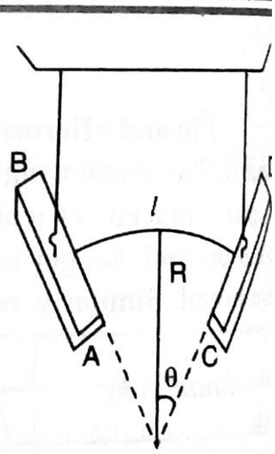

Fig. 1.23 (i)

Let θ be the angular displacement of each bar at any instant during the oscillations. The radius of curvature of the wire at this instant is given by $R = l/2\theta$ [from Fig. 1.23 (i)].

Hence the bending moment of the wire is equal to the couple exerted by it on each bar and is given by

$$\frac{-EAk^2}{R} = -EAk^2\frac{2\theta}{l} \qquad \left(\because \frac{1}{R} = \frac{2\theta}{l}\right)$$

The negative sign indicates that the restoring couple is directed towards the undisturbed position.

Here E = Young's modulus of the material of the wire and

Ak^2 = geometrical moment of inertia of the wire of radius a about the neutral surface.

Now, if I is the M.I. of each bar about the suspension thread (*i.e.*, about an axis passing through its C.G. and perpendicular to its length) and $d^2\theta/dt^2$ is the instantaneous angular acceleration,

$$\text{the couple acting on each bar} = I\frac{d^2\theta}{dt^2}$$

$$\therefore \quad I\frac{d^2\theta}{dt^2} = -EAk^2\,\frac{2\theta}{l} \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \frac{2EAk^2}{Il}\theta = 0.$$

This shows that the motion of each bar is simple harmonic and its time period

$$T_1 = 2\pi\sqrt{\frac{Il}{2\,EAk^2}}$$

$$\text{or} \quad T_1^2 = 4\pi^2\frac{Il}{2\,EAk^2} \quad \text{or} \quad E = \frac{4\pi^2 I\,l}{2Ak^2\,T_1^2}$$

For a wire of circular cross-section of radius a,

$$Ak^2 = \pi a^4/4.$$

$$\therefore \quad E = \frac{8\pi I l}{T_1^2\,a^4}$$

This period of oscillation T_1 is obtained by counting the time for 50 oscillations. The radius of the wire a is measured by a screw gauge. Moment of inertia can be known from the mass and dimensions of the bars. If the bars are cylindrical, of mass M, length L and radius R, then

$$I = M\left[\frac{L^2}{12} + \frac{R^2}{4}\right]$$

To determine G : One of the bars, say AB, is clamped horizontally in a stand and the other bar CD hangs vertically below it at the end of the wire [Fig. 1.23 (*ii*)]. The bar CD is given a slight rotation and released; it executes torsional oscillations about the wire. The period T_2 is found.

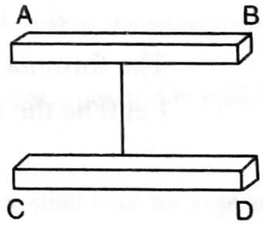

Fig. 1.23 (*ii*)

Then $T_2 = 2\pi\sqrt{I/c}$ where c = couple per unit twist in the wire = $\pi G a^4/2l$, a being the radius of the wire and G the rigidity modulus of the material of the wire. Substituting for c,

$$T_2 = 2\pi\sqrt{\frac{I2l}{\pi G a^4}} \quad \text{or} \quad T_2^2 = 4\pi^2\frac{I\,.\,2l}{\pi G a^4}$$

$$\therefore \quad G = \frac{8\pi I l}{T_2^2\,a^4}.$$

Thus G is calculated.

To determine ν : We know that,

$$\nu = \frac{E}{2G} - 1 = \frac{(8\pi I l/T_1^2 a^4)}{2\,(8\pi I\,l/T_2^2\,a^4)} - 1 = \frac{T_2^2}{2T_1^2} - 1.$$

Substituting the values of T_1 and T_2, the value of ν is obtained.

The advantages of this method are : (*i*) Only a short length of the wire is required for the experiment.

(*ii*) This experiment gives the value of ν in terms of two accurately measurable quantities T_1 and T_2. This eliminates altogether the chief source of error, namely, the measurement of the radius (a) of the wire.

1.23. KÖNIG'S METHOD

The beam is supported on two knife-edges K_1 and K_2 separated by a distance l. Two plane mirrors m_1 and m_2 are fixed near the two ends of the beam at equal distances beyond the knife-edges. [Fig. 1.24 (*a*)]. The two plane mirrors face each other and they are inclined slightly outwards from the vertical.

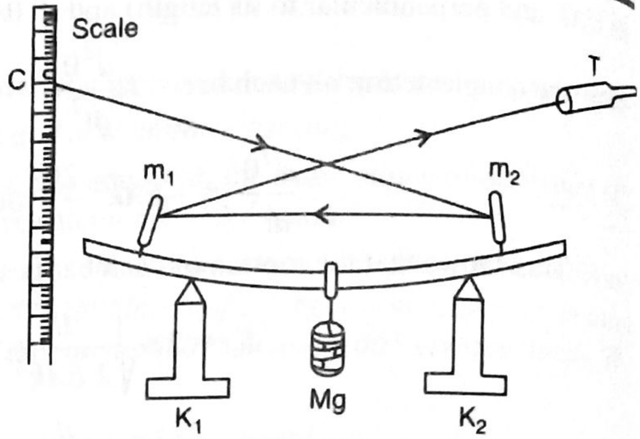

Fig. 1.24 (*a*)

An illuminated translucent scale and a telescope (T) are arranged as shown. The reading of a point C on the scale as reflected first by m_2 and then by m_1 is viewed in the telescope. Let the load suspended at the mid-point of the beam be M. The beam is then bent and the bending is non-uniform. The mirrors at the ends are turned towards each other [Fig. 1.24 (*b*)]. Let the shift in the scale reading be s. The Young's modulus of the material of the beam is then calculated from the relation

$$E = \frac{3Mgl^2\ (2D + L)}{2bd^3 s}$$

where l = Distance between the knife-edges

D = Distance between the scale and the remote mirror, m_2

L = Distance between the two mirrors.

s = Shift in scale reading for a load of M kg

b = Breadth of the beam

d = Thickness of the beam

The formula can be deduced as explained below.

Let θ be the angle through which each end of the beam has been turned due to loading. Then,

$$\theta = \frac{Wl^2}{16\ EAk^2}$$

The mirrors m_1 and m_2 also turn through the same angle θ due to loading. In Fig. 1.24(*b*), m_1 and m_2 represent the initial and m_1' and m_2' the displaced positions of the mirrors. Originally, the image of the scale division at C coincides with the cross-wire and finally when the load is applied, H is seen to be in coincidence with the cross-wire. For convenience in evaluating θ, consider the rays of light to be reversed in their path.

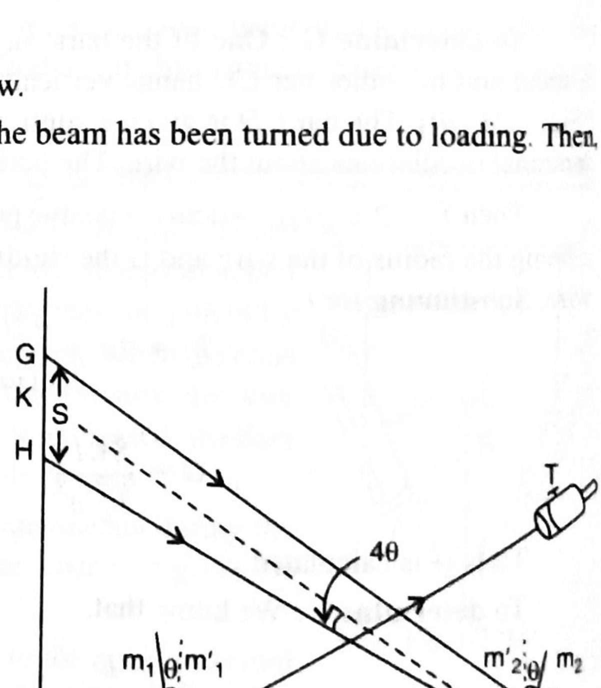

Fig. 1.24 (b)

$TQEC$ will be the original path. When m_1 is turned through an angle θ to the position m_1', QE is turned through 2θ and strikes m_2 at G. Then $EG = L2\theta$. The ray GH is turned through an angle 4θ, since, in addition to QE having moved through 2θ, m_2 itself has turned through θ. Draw GK parallel to EC. Then, $\angle KGH = 4\theta$ and $CK = EG$. $KH = D\ 4\theta$

$\therefore$ The total shift in scale reading $= s = CK + KH$

$$= EG + KH \qquad (\because\ CK = EG)$$

$$= L2\theta + D\,4\theta$$

$$= (L + 2D)\,2\,\theta$$

But $$\theta = \frac{Wl^2}{16\,EAk^2}$$

Hence, $$s = (L + 2D) \times 2 \times \frac{Wl^2}{16\,EAk^2}$$

$$\therefore \quad E = \frac{Wl^2 (L + 2D)}{8\,Ak^2 s}$$

Now $Ak^2 = bd^3/12$ for a beam of rectangular cross-section and

$W = Mg.$

$$\therefore \quad E = \frac{Mgl^2\ (L + 2D)}{8\,(bd^3\,/\,12)s} = \frac{3Mgl^2\ (2D + L)}{2bd^3 s}$$