19UMA06

DIFFERENTIAL EQUATIONS AND LAPLACE TRANSFORM

B.Sc. MATHEMATICS

III - SEMESTER

S. VASUDEVAN
ASSISTANT PROFESSOR
DEPARTMENT OF MATHEMATICS
GOVERNMENT ARTS AND SCIENCE COLLEGE.
KOMARAPALAYAM - 638 183
NAMAKKAL DISTRICT.

CORE VI - DIFFERENTIAL EQUATIONS AND LAPLACE TRANSFORMS

Unit - I :-

Ordinary Differential Equations - second order differential Equations with constant co-efficients - Pariticular Integrals of the form 'v, where v is of the form x, x^2, $\sin ax$, $\cos ax$, $x \sin ax$ and $x \cos ax$.

Unit - II :-

Second order differential Equations with variable co-efficients - both homogeneous linear equations and homogeneous non - linear equations.

Unit - III :-

Partial Differential Equations - Definition - complete solution, singular solution and general solution - solution of equations of standard types $F(p,q) = 0$, $F(x,p,q) = 0$, $F(y,p,q) = 0$, $f(z,p,q) = 0$ and $F_1(x,p) = F(y,q)$ - Clairaut's form - Lagrange's ~~Question~~ equations. $Pp + Qq = R$

Unit - IV :-

Laplace Transforms - Definition - Laplace transforms of standard functions - Elementary theorems - Problems.

Unit - V :-

Inverse Transforms - Definition - Laplace standard formulae - Elementary Theorems - Applications ~~of~~ to second ~~on~~ order linear differential equations [Problems With only one differential equation]

UNIT-I

Ordinary Differential Equations

- Second order differential Equations with constant co-efficients - Pariticular Integrals of the form 'v, where v is of the form x, x^2, $\sin ax$, $\cos ax$, $x \sin ax$ and $x \cos ax$.

Unit - I :-

Higher Order Linear Differential Equations with constant co-efficients

General form of a linear differential equation of the n^{th} order with constant co-efficient is

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = x \rightarrow (1)$$

Where $k_1, k_2 \dots k_n$ are constants.

Such equations are most important in the study of electro - mechanical vibrations and other engineering problems.

In discussing linear equations with co-efficients, it will be convenient to denote the equations $\frac{d}{dx}$ by a single letter D. Thus D is the differential operator so that

$Dy = \frac{dy}{dx}$, similarly $D^2 y = \frac{d^2 y}{dx^2}$, $D^3 y = \frac{d^3 y}{dx^3}$ etc ...

Generally, $D^n y = \frac{d^n y}{dx^n}$

The equation (1) above can be written in the symbotic form

$(D^n + k_1 D^{n-1} + \dots + k_n) y = x$ ie, $f(D) y = x$

where $f(D) = D^n + k_1 D^{n-1} + \dots + k_n$

a polynomial in D.

Note :-

1). $\frac{1}{D} x = \int x \, dx$

2). $\frac{1}{D-a} x = e^{ax} \int x e^{-ax} dx$

3). $\frac{1}{D+a} x = e^{-ax} \int x e^{ax} dx$

(i) The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P and Q are constants and R is a function of x or constant.

(ii) Differential operators

The symbol D stands for the operation of differential

(ie.,) $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$

$\frac{1}{D}$ stands for the operation of integration

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ canbe written in the operator form

$$D^2y + PDy + Qy = R \text{ (or) } (D^2+PD+Q)y = R$$

(iii) Complete solution is

y = complementary function + Particular Integral

(iv) To find the complementary functions

	Roots of A.E.	C.F.
1.	Roots are Real and different m_1, m_2 $(m_1 \neq m_2)$	$Ae^{m_1 x} + Be^{m_2 x}$
2	Roots are Real and equal $m_1 = m_2 = m$ (say)	$(Ax+B)e^{mx}$ (or) $(A+Bx)e^{mx}$
3.	Roots are imaginary $\alpha + i\beta$	$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

(v) To find the particular integral :-

$$P.I = \frac{1}{f(D)} X$$

	X	P.I
1.	e^{ax}	$P.I = \frac{1}{f(D)} e^{ax} = e^{ax} \frac{1}{f(a)}, \; f(a) \neq 0$ $= x e^{ax} \frac{1}{f'(a)}, \; f(a) = 0, \; f'(a) \neq 0$ $= x^2 e^{ax} \frac{1}{f''(a)}, \; f(a) = 0, f'(a) = 0, \; f''(a) \neq 0$
2.	x^n	$P.I = \frac{1}{f(D)} x^n = [[f(D)]^{-1}] x^n$ Expand $[f(D)]^{-1}$ and then operate
3.	$\sin ax$ (or) $\cos ax$	$P.I = \frac{1}{f(D)} [\cos ax \text{ (or) } \sin ax]$ Replace D^2 by $-a^2$
4.	$e^{ax} \phi(x)$	$P.I = \frac{1}{f(D)} e^{ax} \phi(x)$ $= e^{ax} \frac{1}{f(D+a)} \phi x$

Result :-

(i) $\frac{1}{D-a}\phi(x) = e^{ax}\int e^{-ax}\phi(x)\,dx$

(ii) $\frac{1}{D+a}\phi(x) = e^{-ax}\int e^{ax}\phi(x)\,dx$

6
-5
-2 -3

1. Solve $(D^2 - 5D + 6)y = 0$

Soln :-

Given $(D^2 - 5D + 6)y = 0$

The auxiliary equation is $m^2 - 5m + 6 = 0$

(i.e.,) $(m-3)(m-2) = 0$

(i.e.,) $m = 2, m = 3$

$\therefore C.F = Ae^{2x} + Be^{2x}$

$\therefore$ The general solution is given by $y = C.F$

i.e., $y = Ae^{2x} + Be^{3x}$

---

2. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$

Soln :-

Given $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$

$(D^2 - 6D + 13)y = 0$

The auxillary equation is $m^2 - 6m + 13 = 0$

i.e., $m = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$

$= \frac{6 \pm 4i}{2} = 3 \pm 2i$

Hence, the solution is $y = e^{3x}[A\cos(2x) + B\sin(2x)]$.

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow m = \frac{6 \pm \sqrt{36-52}}{2}$.

3. Solve $(D^2+1)y=0$, given $y(0)=0$, $y'(0)=1$.

Soln:-

Given : $(D^2+1)y=0$

The auxiliary equation is $m^2+1=0 \Rightarrow m^2=-1$

$m=\pm i$

$y = A\cos x + B\sin x$

ie., $y(x) = A\cos x + B\sin x$

$\sin 0 = 0$

$\cos 0 = 1$

given : $y(0)=0 \Rightarrow y(0) = A = 0$

$y'(x) = -A\sin x + B\cos x$

given : $y'(0)=1 \Rightarrow y'(0) = B = 1$

$\therefore (1) \Rightarrow y(x) = \sin x$

4. Solve $(D^2+1)y=0$ given $y(0)=1$, $y(\pi/2)=0$

Soln:-

Given : $(D^2+1)y=0$

The auxiliary equation is $m^2+1=0 \Rightarrow m^2=-1$

$m=\pm i$

$y = A\cos x + B\sin x$

ie, $y(x) = A\cos x + B\sin x$

Given : $y(0)=1 \Rightarrow y(0)=A=1$

given : $y(\pi/2)=0 \Rightarrow y(\pi/2)=B=0$

$(1) \Rightarrow y(x) = \cos x$

Type - I

B. Problems Based on $P.I = \frac{1}{f(D)} e^{ax}$

Replace D by a.

5. Solve $(D^2-4D+13)y = e^{2x}$

Soln:-

Given : $(D^2-4D+13)y = e^{2x}$

The auxiliary equation is $m^2-4m+13=0$

$$m = \frac{4\pm\sqrt{16-52}}{2} = \frac{4\pm\sqrt{-36}}{2} = \frac{4\pm 6i}{2} = 2\pm 3i$$

$$\therefore C.F = e^{2x}(A\cos 3x + B\sin 3x)$$

$$P.I = \frac{1}{D^2-4D+13}e^{2x} = \frac{1}{4-8+13}e^{2x}$$ Replace D by 2

$$= \frac{1}{9}e^{2x}$$

$$y = C.F + P.I$$

$$y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{1}{9}e^{2x}.$$

---

6. Solve $y'' - 3y' + 2y = e^{x} - e^{2x}$

Soln:-

Given : $y'' - 3y' + 2y = e^{x} - e^{2x}$

$(D^2-3D+2)y = e^{x} - e^{2x}$

$D^2-3D+2 = e^{x} - e^{2x}$

The auxiliary equation is $D^2-3D+2=0$

$(m-1)(m-2)=0$

$m = 1, 2$

$\therefore C.F = Ae^{x} + Be^{2x}$

$$P.I_{1} = \frac{1}{D^2-3D+2}e^{x}$$

$$= \frac{1}{1-3+2}e^{x} = \frac{1}{0}e^{x} = x\frac{1}{2D-3}e^{x}$$

$$= x \cdot \frac{1}{2-3} e^{x} = -xe^{x}$$

$$P.I_2 = \frac{1}{D^2-3D+2}(-e^{2x})$$

$$= -\frac{1}{4-6+2} e^{2x} = \frac{1}{0} e^{2x} = -x \frac{1}{2D-3} e^{2x}$$

$$= -x \frac{1}{4-3} e^{2x} = -xe^{2x}$$

$$\therefore P.I = P.I_1 + P.I_2$$

$$P.I = -xe^{x} - xe^{2x}$$

$$= -x\left[e^{x}+e^{2x}\right]$$

$$\boxed{y = Ae^{x} + Be^{2x} - x(e^{x}+e^{2x}).}$$

7. Solve $(4D^2-4D+1)y = 4$.

soln:-

Given : $(4D^2-4D+1)y = 4$

The auxiliary equation is $4m^2-4m+1=0$

4

-4

$\frac{-2}{4}$ 2 $\frac{-2}{4}$

$$(2m-1)(2m-1)=0$$

$$(2m-1)^2=0 \Rightarrow m = 1/2, 1/2$$

$$P.I = \frac{1}{4D^2-4D+1} 4e^{0x} = \frac{4}{1} e^{0x} = 4$$

$$\therefore y = C.F + P.I$$

$$\boxed{y = (Ax+B)e^{\frac{1}{2}x} + 4}$$

8. Solve $(D^2-4)y = e^{2x} + e^{-4x}$

Soln:-

Given :- $(D^2-4)y = e^{2x} + e^{-4x}$

The auxiliary equation is $m^2-4=0$

$$(m-2)(m+2)=0$$

$$m = 2, -2$$

$$\therefore C.F = Ae^{2x} + Be^{-2x}$$

$$P.I_1 = \frac{1}{D^2-4}e^{2x} = \frac{1}{4-4}e^{2x} = \frac{1}{0}e^{2x}$$

$$= x\frac{1}{2D}e^{2x} = x\,\tfrac{1}{4}\,e^{2x} = \frac{x}{4}e^{2x}$$

$$P.I_2 = \frac{1}{D^2-4}e^{-4x} = \frac{1}{16-4}e^{-4x} = \frac{1}{12}e^{-4x}$$

$$\therefore y = C.F + P.I_1 + P.I_2$$

$$\boxed{y = Ae^{2x} + Be^{-2x} + \frac{x}{4}e^{2x} + \frac{1}{12}e^{-4x}}$$

---

9. Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$

Soln :-

Given :- $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$

$$(D^2+4D+4)y = e^{-2x}$$

The ~~aux~~ auxiliary equation is $m^2+4m+4=0$

$$(m+2)^2=0$$

$$m = -2, -2$$

$$C.F = (Ax+B)e^{-2x}$$

$$P.I = \frac{1}{D^2+4D+4} e^{-2x} = \frac{1}{4-8+4} e^{-2x}$$

$$= \frac{1}{0} e^{-2x} = x \frac{1}{2D+4} e^{-2x}$$

$$= x \frac{1}{-4+4} e^{-2x} = \frac{x}{0} e^{-2x}$$

$$= x^2 \frac{1}{2} e^{-2x}$$

$$= \frac{x^2}{2} e^{-2x}$$

$\therefore y = C.F + P.I$

$$\boxed{y = (Ax+B)e^{-2x} + \frac{x^2}{2} e^{-2x}.}$$

Type-2 PROBLEMS BASED ON $P.I = \frac{1}{f(D)} \sin ax$ (or) $\frac{1}{f(D)} \cos ax$ $\Rightarrow$ Replace D^2 by $-a^2$.

10. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin 3x$

Soln:-

Given $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin 3x$

(ie) $(D^2 + 3D + 2)y = \sin 3x$

The auxiliary equation is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -2, -1 \Rightarrow m = -1, -2$

2, 3, 1, 2

$C.F = Ae^{-x} + Be^{-2x}$

$$P.I = \frac{1}{D^2+3D+2} \sin 3x$$

$$= \frac{1}{-9+3D+2} \sin 3x$$ Replace D^2 by $-a^2$

$$= \frac{1}{3D-7}\sin 3x = \frac{1}{3D-7}\,\frac{3D+7}{3D+7}\sin 3x$$

$$= \frac{3D+7}{3D^2-7^2}\sin 3x = \frac{3D+7}{9(-9)-49}\sin 3x$$

$$= \frac{3D+7}{-81-49}\sin 3x = \frac{3D+7}{-130}\sin 3x$$

$$= -\frac{1}{130}\left[3D(\sin 3x) + 7\sin 3x\right]$$

$$= -\frac{1}{130}\left[9\cos 3x + 7\sin 3x\right]$$

$$\therefore P.I = \frac{-1}{130}\left[9\cos 3x + 7\sin 3x\right]$$

$$\therefore y = C.F + P.I$$

$$\boxed{y = Ae^{-x} + Be^{-2x} - \frac{1}{130}(9\cos 3x + 7\sin 3x)}$$

---

11. Solve $(D^2+4)y = \cos 2x$

Soln:-

Given that $(D^2+4)y = \cos 2x$

The auxiliary equation is $m^2+4=0$

$$m^2+4=0$$

$$m^2=-4$$

$$m=\sqrt{-4}$$

$$m=\pm 2i$$

$$C.F = A\cos 2x + B\sin 2x$$

$$P.I = \frac{1}{D^2+4}\cos 2x$$

$$= \frac{1}{-4+4}\cos 2x = \frac{1}{0}\cos 2x$$

$$= x\frac{1}{2D}\cos 2x \qquad \text{Replace } D^2 \text{ by } -a^2$$

$$= \frac{x}{2}\int \cos 2x\,dx = x/2\,\frac{\sin 2x}{2} = \frac{x}{4}\sin 2x$$

$y = C.F + P.I$

$$\boxed{y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x}$$

12. Solve $(D^2+1)y = \sin x$

Soln:-

Given that $(D^2+1)y = \sin x$

The auxiliary equation is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i$$

$$C.F = A\cos x + B\sin x$$

$$P.I = \frac{1}{D^2+1}\sin x$$

$$= \frac{1}{-1+1}\sin x = \frac{1}{0}\sin x = x\frac{1}{2D}\sin x$$

$$= \frac{x}{2}\int \sin x\,dx = \frac{x}{2}(-\cos x)$$

$$P.I = -x/2\cos x$$

$y = C.F + P.I$

$$\boxed{y = A\cos x + B\sin x - x/2\cos x.}$$

13. Solve $(D^2+1)y = \sin x \sin 2x$

Soln:-

Given that $(D^2+1)y = \sin x \sin 2x$

The auxiliary equation is $m^2+1=0$

$$m^2 = -1$$
$$m = \pm i$$

$C.F = A\cos x + B\sin x$

$(D^2+1)y = \sin 2x . \sin x$

$= -\frac{1}{2}[\cos(3x) - \cos x] = -\frac{1}{2}\cos 3x + \frac{1}{2}\cos x$

$P.I_1 = \frac{1}{D^2+1}\left[-\frac{1}{2}\cos 3x\right]$

$= \frac{-1}{2}\left(\frac{1}{-9+1}\right)\cos 3x$ Replace D^2 by $-a^2$

$P.I_1 = \frac{1}{16}\cos 3x$

$P.I_2 = \frac{1}{D^2+1}\left[\frac{1}{2}\cos x\right]$ Replace D^2 by $-a^2$

$= \frac{1}{2}\left[\frac{1}{-1+1}\right]\cos x$

$= \frac{1}{2} x \frac{1}{2D}\cos x$

$= \frac{x}{4}\int \cos x \, dx = \frac{x}{4}\sin x$

$P.I_2 = \frac{x}{4}\sin x$

$P.I = \frac{1}{16}\cos 3x + \frac{x}{4}\sin x$

$$\boxed{y = A\cos x + B\sin x + \frac{1}{16}\cos 3x + \frac{x}{4}\sin x}$$

I(i) GENERAL ODE PROBLEMS

14. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$

Solns:-

Given that $\left(\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2\right.$

$$(D^2 - 6D + 9)\, y = 6e^{3x} + 7e^{-2x} - \log 2$$

The auxiliary equation is $m^2 - 6m + 9 = 0$

$$(m-3)(m-3) = 0$$

9, -6, -3, -3

$$m = 3, 3.$$

$$C.F = (Ax + B)\, e^{3x}$$

$$(D^2 - 6D + 9)\, y = 6e^{3x} + 7e^{-2x} - \log 2$$

$$P.I_1 = \frac{1}{D^2 - 6D + 9} 6e^{3x} = \frac{1}{(D-3)^2} 6e^{3x}$$

$$= 6 \frac{1}{(3-3)^2} e^{3x}$$

$$= x \frac{1}{2(D-3)} 6e^{3x}$$

$$= \frac{6x}{2} \frac{1}{3-3} e^{3x} = x^2 \frac{1}{2} 6e^{3x}$$

$$P.I_1 = 3x^2 e^{3x}$$

$$P.I_2 = \frac{1}{(D-3)^2} 7e^{-2x} = 7 \frac{1}{(-2-3)^2} e^{-2x}$$

$$= 7 \frac{1}{(-5)^2} e^{-2x} = 7 \frac{1}{25} e^{-2x} = \frac{7}{25} e^{-2x}$$

$$P.I_2 = \frac{7}{25} e^{-2x}$$

$$P.I_3 = \frac{1}{(D-3)^2} (+\log 2) \cdot e^{0x}$$

$$P.I_3 = \frac{1}{9} \log 2 = \frac{\log 2}{9}$$

$$y = C.F + P.I$$

$$\boxed{y = (Ax+B)e^{3x} + 3x^2 e^{3x} + \frac{7}{25}e^{-2x} - \frac{\log 2}{9}}$$

Type : 3

Suppose $F(x) = x^m$

Here $\frac{1}{\phi(x)} x^m = [\phi(D)]^{-1} x^m$

The Binomial expansions are

$$(1-D)^{-1} = 1 + D + D^2 + \cdots \infty$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots \infty$$

$$(1-D)^{-2} = 1 + 2D + 3D^2 + \cdots \infty$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - \cdots \infty$$

15). Solve : $(D^2+D+1)y = x$

Soln:-

Given that $(D^2+D+1)y = x$

The auxiliary equation is $m^2+m+1 = 0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\therefore m = -1/2 \pm \frac{i\sqrt{3}}{4}$$

$$C.F = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

$$C.F = e^{-1/2x}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I = \frac{1}{D^2+D+1}x$$

$$= (D^2+D+1)^{-1}x \quad = [1+D+D^2]^{-1}x$$

$$= [1-(D+D^2)+(D+D^2)^2 - \cdots\cdots]x$$

$$= [1-D]x$$

$$= x - Dx$$

$$P.I = x-1$$

The general equation is

$$y = C.F + P.I$$

$$\boxed{y = e^{-1/2x}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] + x - 1}$$

---

16. Solve : $(D^2+4)y = x^2$

Soln:- Given that $(D^2+4)y = x^2$

The auxiliary equation is $m^2+4=0$

$$m^2 = -4$$

$$m = \pm 2i$$

$$m = \pm 2i$$

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$C.F = e^{0x}(A\cos 2x + B\sin 2x)$$

$$C.F = A\cos 2x + B\sin 2x$$

$$P.I = \frac{1}{(D^2+4)}x^2$$

$$= \frac{1}{4+D^2} x^2$$

$$= \frac{1}{4}\left[1+\frac{D^2}{4}\right]^{-1} x^2 = \frac{1}{4}\left[1-\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+\cdots\right]x^2$$

$$= \frac{1}{4}\left[x^2 - D^2\frac{(x^2)}{4}\right]$$

$$= \frac{1}{4}\left[x^2 - 2/4\right]$$

$$= \frac{1}{4}\left[\frac{x^2}{4} - \frac{2}{16}\right] = \frac{x^2}{4} - 1/8.$$

$$P.I = \frac{x^2}{4} - \frac{1}{8}$$

The general equation is

$$y = C.F + P.I$$

$$\boxed{y = (A\cos 2x + B\sin 2x + \frac{x^2}{4} - 1/8}$$

---

(17). Solve : $(D^2 - 2D + 1)y = x^2 + 1 + \sin 2x$

soln:-

Given that $(D^2-2D+1)y = x^2+1+\sin 2x$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$\therefore$ The roots are real and equal

$$C.F = (Ax+B)e^x.$$

$$P.I = \frac{1}{D^2-2D+1}(x^2+1+\sin 2x)$$

$$P.I = \frac{1}{(D-1)^2}(x^2+1) + \frac{1}{(D-1)^2}\sin 2x$$

$$P.I_1 = \frac{1}{(D-1)^2}(x^2+1)$$

$$= \frac{1}{[(1-D)]^2}(x^2+1) = \frac{1}{(1-D)^2}(x^2+1)$$

$$= (1-D)^{-2}(x^2+1)$$

$$= (1+2D+3D^2+\dots)(x^2+1)$$

$$\cancel{= x^2+2Dx^2+3D^2x^2+1+2D+3D^2}$$

$$\cancel{= x^2+4x+2x+1}$$

$$= (x^2+1)+2D(x^2+1)+3D^2(x^2+1)$$

$$P.I_1 = x^2+1+4x+0+6 = x^2+4x+7$$

$$P.I_2 = \frac{1}{(D-1)^2}\sin 2x = \frac{1}{D^2-2D+1}\sin 2x$$

$$= \frac{1}{-4-2D+1}\sin 2x$$

$$= \frac{1}{-2D-3}\sin 2x = \frac{1}{-(2D+3)}\sin 2x$$

$$= \frac{-1\,(2D-3)}{(2D+3)(2D-3)}\sin 2x$$

$$= \frac{-(2D-3)}{4D^2-9}\sin 2x = \frac{-(2D-3)}{4(-4)-9}\sin 2x$$

$$= \frac{-(2D-3)}{-25}\sin 2x$$

$$= \frac{2D-3}{25}\sin 2x$$

$$P.I_2 = \frac{2D(\sin 2x) - 3\sin 2x}{25}$$

$$= \frac{4\cos 2x - 3\sin 2x}{25}.$$

$\therefore$ The general solution is

$$\boxed{y = (Ax+B)e^x + x^2+4x+7+\frac{1}{25}(4\cos 2x - 3\sin 2x)}$$

(18). Solve :- $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x^3 - 3x^2 + 1$

Soln :-

Given equation is $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x^3 - 3x^2 + 1$

$$(D^2 - D + 1)y = x^3 - 3x^2 + 1$$

The auxiliary equation is $D^2 - D + 1 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-1) \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

$$m = \frac{1 \pm i\sqrt{3}}{2}$$

$$m = 1/2 \; ; \; m = i\sqrt{3}/2$$

The roots are imaginary

$$y = e^{x/2}\left[A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2} x\right]$$

$$P.I = \frac{1}{(D^2 - D + 1)}(x^3 - 3x^2 + 1)$$

$$= \frac{1}{[1 - (D - D^2)]}(x^3 - 3x^2 + 1)$$

$$= [1 - (D - D^2)]^{-1}(x^3 - 3x^2 + 1)$$

$$= [1 + (D - D^2) + (D - D^2)^2 + (D - D^2)^3 + \cdots](x^3 - 3x^2 + 1)$$

$$= [1 + D - D^2 - 2D^3 + D^2 + D^3][x^3 - 3x^2 + 1]$$

$$= [1 + D - D^3]\; x^3 - 3x^2 + 1$$

$$= x^3 - 3x^2 + 1 + D(x^3 - 3x^2 + 1) - D^3(x^3 - 3x^2 + 1)$$

$$= x^3 - \cancel{3x^2} + 1 + \cancel{3x^2} - 6x - 6$$

$$P.I = x^3 - 6x - 5$$

$$\boxed{y = e^{x/2}\left[A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x\right] + (x^3 - 6x - 5)}$$

(19). Solve $(D^2+3D+2)y = e^{2x}+x^2+\sin x$

Soln:-

Given that equation $(D^2+3D+2)y = e^{2x}+x^2+\sin x$

The auxiliary equation is $m^2+3m+2=0$

$(m+1)(m+2)=0$

$m=-1,-2$

The roots are real and distinct

$$C.F = Ae^{-x}+Be^{-2x}$$

$$P.I = \frac{1}{D^2+3D+2}(e^{2x}+x^2+\sin x)$$

$$P.I_1 = \frac{1}{D^2+3D+2}e^{2x} = \frac{1}{2^2+6+2}e^{2x}$$

$$= \frac{1}{12}e^{2x}$$

Using with Bionomal Expansions formula:-

$$P.I_2 = \frac{1}{D^2+3D+2}x^2 = \frac{1}{2+3D+D^2}x^2$$

$$= \frac{1}{2\left(1+\frac{3D+D^2}{2}\right)}x^2 = \frac{1}{2}\left[1+\frac{3D+D^2}{2}\right]^{-1}x^2$$

$$= \frac{1}{2}\left[1-\left(\frac{3D+D^2}{2}\right)+\left(\frac{3D+D^2}{2}\right)^2-\cdots\infty\right]x^2$$

$$= \frac{1}{2}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\frac{1}{4}\left[9D^2+6D^3+D^4\right]-\cdots\infty\right]x^2$$

$$= \frac{1}{2}\left[1-\frac{1}{2}(3D+D^2)+\frac{1}{4}(9D^2)-\cdots\infty\right]x^2$$

$$= \frac{1}{2}\left[x^2-\frac{1}{2}(3D+D^2)x^2+\frac{1}{4}(9D^2)x^2\right]$$

$$= \frac{1}{2}\left[x^2-\frac{1}{2}\left[3\times 2x+2\right]+\frac{1}{4}\left[9\times 2\right]\right]$$

$$= \frac{1}{2}\left[x^2-\frac{\cancel{6}^{3}x}{\cancel{2}}-\cancel{2}/\cancel{2}+\cancel{18}^{9}/\cancel{4}_{2}\right]$$

$$= \frac{1}{2}\left[x^2 - 6x - 1 + \frac{9}{2}\right]$$

$$P.I_2 = \frac{1}{2}\left[x^2 - 6x + \frac{9}{2} - 1\right] = \frac{1}{2}\left[x^2 - 6x + \frac{7}{2}\right]$$

$$P.I_3 = \frac{1}{D^2 + 3D + 2} \sin x$$

$$= \frac{1}{-1 + 3D + 2} \sin x = \frac{1}{3D+1} \sin x$$

$$= \frac{3D-1}{(3D+1)(3D-1)} \sin x$$

$$= \frac{3D-1}{9D^2 - 1} \sin x$$

$$= \frac{3D-1}{-9-1} \sin x = \frac{3D-1}{-10} \sin x$$

$$= \frac{3D \sin x - \sin x}{-10} = \frac{3\cos x - \sin x}{-10}$$

$$P.I_3 = -\frac{1}{10}(3\cos x - \sin x).$$

The general solution is

$$y = C.F + P.I$$

$$y = Ae^{-x} + Be^{-2x} + \frac{1}{12}e^{2x} + \frac{1}{2}\left[x^2 - 6x + \frac{7}{2}\right] - \frac{1}{10}(3\cos x - \sin x).$$

$$\boxed{y = Ae^{-x} + Be^{-2x} + \frac{1}{12}e^{2x} + \frac{1}{2}\left(x^2 - 6x + \frac{7}{2}\right) - \frac{1}{10}(3\cos x - \sin x)}$$ A.

Type : 4

$X = e^{ax} \cdot V(x)$, where V is of the form $\sin \beta x$, $\cos \beta x$ or x^m.

$$P.I = \frac{1}{f(D)} e^{ax} V(x) = e^{ax} \cdot \frac{1}{f\left(\frac{1}{(D+\alpha)}\right)} V(x)$$

$\frac{1}{f(D+\alpha)} V(x)$ is evaluated by using the rule (3) (or) (4).

Note:- This rule is referred to as Exponential shift rule.

20. Solve : $(D^2 - 4D + 3)\, y = x^3 e^{2x}$.

Soln:-

Given that equation $(D^2 - 4D + 3) y = x^3 e^{2x}$.

The auxiliary equation $m^2 - 4m + 3 = 0$

3, -4, -1, -3

$$(m-1)(m-3) = 0$$

$$m = 1, 3$$

The complementary function is

$$C.F = Ae^{x} + Be^{3x}$$

The particular integral is

$$P.I = \frac{1}{D^2 - 4D + 3} x^3 e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 3} x^3$$

$$= e^{2x} \frac{1}{D^2 + \cancel{4D} + \cancel{4} - \cancel{4D} - 8 + 3} x^3$$

$$= e^{2x} \frac{1}{D^2 - 1} x^3$$

$$= e^{2x} \frac{1}{-(1-D^2)} x^3 \quad = -e^{2x} [1 - D^2]^{-1} x^3$$

$$= -e^{2x} [1 + D^2 + (D^2)^2 + \cdots] x^3$$

$$= -e^{2x} [1 + D^2] x^3 \quad = -e^{2x} [x^3 + D^2 x^3]$$

P.I $= -e^{2x} [x^3 + 6]$

The general solution is

$$y = C.F + P.I$$

$$\boxed{y = Ae^{x} + Be^{3x} - e^{2x} [x^3 + 6]_{u}}$$

21. Solve :- $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{x} \cos x$

Soln :-

Given that equation is

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{x} \cos x$$

$$(D^2 - 5D - 6)y = e^{x} \cos x$$

The auxiliary equation is

$$\cancel{(D^2 - 5D - 6)y = 0}$$

$$\cancel{m^2 - 5m + 6 = 0}$$

$$\cancel{(m+2)(m-6) = 0}$$

$$\cancel{m = 6, -1}$$

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3.$$

$$C.F = Ae^{2x} + Be^{3x}$$

The P.I is

$$P.I = \frac{1}{D^2-5D+6} e^x \cos x$$

$$P.I = e^x \frac{1}{(D+1)^2 - 5(D+1) + 6} \cos x$$

$$= e^x \frac{1}{D^2+2D+1-5D-5+6} \cos x$$

$$= e^x \frac{1}{D^2-3D+2} \cos x$$

$$= e^x \frac{1}{-1-3D+2} \cos x = e^x \frac{1}{-3D+1} \cos x$$

$$= e^x \frac{1}{-(3D-1)} \cos x$$

$$= -e^x \frac{(3D+1)}{(3D-1)(3D+1)} \cos x$$

$$= -e^x \frac{(3D+1)}{9D^2-1} \cos x$$

$$= -e^x \frac{3D+1}{-9-1} \cos x = -e^x \frac{3D+1}{-10} \cos x$$

$$= \frac{e^x}{10} [3D\cos x + \cos x]$$

$$= \frac{e^x}{10} [-3\sin x + \cos x]$$

$$P.I = \frac{e^x}{10} [\cos x - 3\sin x]$$

The general equation is

$$y = C.F + P.I$$

$$\boxed{y = Ae^{2x} + Be^{3x} + \frac{e^x}{10} (\cos x - 3\sin x)}$$

Type : 5

$X = x \cdot V(x)$, where $V(x)$ is of the form $\sin \alpha x$ (or) $\cos \alpha x$.

$$P.I = \frac{1}{f(D)} x V(x) = x \cdot \frac{1}{f(D)} V(x) + \frac{d}{dD}\left\{\frac{1}{f(D)}\right\} V(x)$$

$$= x \cdot \frac{1}{f(D)} V(x) - \frac{f'(D)}{\{f(D)\}^2} V(x)$$

By repeated applications of this rule, we can find the P.I. When $X = x^r V(x)$, where r is a positive integer.

1. Solve : $\frac{d^2y}{dx^2} + 4y = x \sin x$

Solns:-

Given that equation is $\frac{d^2y}{dx^2} + 4y = x \sin x$

The auxiliary equation is $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

$$C.F = A \cos 2x + B \sin 2x$$

$$P.I = \frac{1}{(D^2+4)} x \sin x$$

$$= \left[x - \frac{1}{\phi(D)} \phi'(D)\right] \frac{1}{\phi(D)} V$$

$$= \left[x - \frac{1}{D^2+4}(2D)\right] \frac{1}{D^2+4} \sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{D^2+4} \sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{-1+4} \sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{3} \sin x$$

$$= \frac{1}{3}\left[x - \frac{2D}{D^2+4}\right] \sin x$$

$$= \frac{1}{3}\left[x \sin x - \frac{2D \sin x}{D^2+4}\right]$$

$$= \frac{1}{3}\left[x \sin x - \frac{2D(\sin x)}{-1+4}\right]$$

$$= \frac{1}{3}\left[x \sin x - \frac{2D \sin x}{3}\right]$$

$$= \frac{1}{3}\left[x \sin x - \frac{2\cos x}{3}\right]$$

$$P.I = \frac{1}{3} x \sin x - \frac{2\cos x}{9}$$

The general equation is

$$y = C.F + P.I$$

$$y = A \cos 2x + B \sin 2x + \frac{1}{3} x \sin x - \frac{2\cos x}{9}$$

Ans.

22. Solve :- $(D^2-2D+1)y = xe^x \sin x$

(2)

Soln :-

Given that equation $(D^2-2D+1)y = xe^x \sin x$

The auxiliary equation is $m^2-2m+1=0$

$$(m-1)(m-1)=0$$

$$m=1,1$$

$$C.F = e^x(Ax+B).$$

The particular integral is

$$P.I = \frac{1}{D^2-2D+1}(xe^x \sin x)$$

$$= \frac{1}{D^2-2D+1} e^x x \sin x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+1} \cancel{e^x}\, x \sin x$$

$$= e^x \frac{1}{D^2+\cancel{2D}+\cancel{1}-\cancel{2D}-\cancel{2}+\cancel{1}} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \left[x - \frac{1}{D^2} 2D\right] \frac{1}{D^2} \sin x$$

$$= e^x \left[x - 2/D\right] \frac{1}{D^2} \sin x$$

$$= e^x \left[x - 2/D\right] \frac{1}{-1} \sin x$$

$$= -e^x \left[x \sin x - \frac{2}{D} \sin x\right]$$

$$P.I = -e^x [x \sin x + 2\cos x]$$

$$y = C.F + P.I$$

$$\boxed{y = e^x(Ax+B) - e^x(x \sin x + 2\cos x)}$$

(23). Solve : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

(3) Soln:- Given that equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x.$$

$$(D^2 - 3D - 2)y = xe^{3x} + \sin 2x$$

The auxiliary equation is $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$C.F = Ae^{x} + Be^{2x}.$$

$$P.I_1 = \frac{1}{D^2 - 3D + 2} x e^{3x}$$

$$= e^{3x} \frac{1}{(D+3)^2 - 3(D+3) + 2} x = e^{3x} \frac{1}{D^2 + 6D + 9 - 3D - 9 + 2} x$$

$$= e^{3x} \frac{1}{D^2 + 3D + 2} x = e^{3x} \frac{1}{2 + 3D + D^2} x$$

$$= e^{3x} \frac{1}{2\left[1 + \frac{3D}{2} + \frac{D^2}{2}\right]} x$$

$$= e^{3x}/2 \left[1 + \frac{3D + D^2}{2}\right]^{-1} x$$

$$= \frac{e^{3x}}{2}\left[1 - \frac{3D + D^2}{2} + \cdots\right] x$$

$$= \frac{e^{3x}}{2}\left[1 - \frac{3D + D^2}{2}\right] x = \frac{e^{3x}}{2}\left[x - \frac{3D(x)}{2}\right]$$

$$P.I_1 = \frac{e^{3x}}{2}\left[x - 3/2\right] = \frac{xe^{3x}}{2} - \frac{3e^{3x}}{4}$$

$$P.I_2 = \frac{1}{D^2 - 3D + 2} \sin 2x$$

$$= \frac{1}{-4 + 3D + 2} \sin 2x = \frac{1}{-3D - 2} \sin 2x$$

$$= \frac{1}{-(3D+2)} \sin 2x$$

$$= -\frac{(3D-2)}{(3D+2)(3D-2)} \sin 2x$$

$$= -\frac{3D-2}{9D^2-4} \sin 2x$$

$$= -\frac{3D-2}{-36-4} \sin 2x = -\frac{3D-2}{-40} \sin 2x$$

$$= \frac{3D-2}{40} \sin 2x = \frac{3D \sin 2x - 2 \sin 2x}{40}$$

$$P.I_2 = \frac{6\cos 2x - 2\sin 2x}{40} = \frac{3\cos 2x - \sin 2x}{20}$$

$$y = C.F + P.I$$

$$\boxed{y = Ae^{x} + Be^{2x} + \frac{xe^{3x}}{2} - \frac{3e^{3x}}{4} + \frac{3\cos 2x - \sin 2x}{20}}$$

(24) (A) Solve $(D^2+2D+5) = xe^{x}$

Soln:-

Given that equation is $(D^2+2D+5) = xe^{x}$

The auxiliary equation is $m^2+2m+5=0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$m = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$\therefore C.F = e^{-x}(A\cos 2x + B \sin 2x)$$

The particular equation is

$$P.I = \frac{1}{(D^2+2D+5)} xe^x$$

$$= e^x \frac{1}{(D+1)^2+2(D+1)+5} x$$

$$= e^x \frac{1}{D^2+2D+1+2D+2+5} x$$

$$= e^x \frac{1}{D^2+4D+8} x$$

$$= e^x \frac{1}{8+4D+D^2} x$$

$$= e^x \frac{1}{8}\left[\frac{1}{1+\frac{4D+D^2}{8}}\right] x$$

$$= e^x \frac{1}{8}\left[1+\frac{4D+D^2}{8}\right]^{-1} x$$

$$= \frac{e^x}{8}\left[1-\frac{4D}{8}\right] x$$

$$= \frac{e^x}{8}\left[x-\frac{4Dx}{8}\right] = \frac{e^x}{8}\left[x-4/8\right]$$

$$P.I = \frac{e^x}{8}\left[x-1/2\right]$$

$$\boxed{y = e^{-x}(A\cos 2x + B\sin 2x) + \frac{e^x}{8}(x-1/2)}$$

---

25. Solve $(3D^2+D-14)y = 13e^{2x}$

(5)

Soln:-

Given that equation is

$$(3D^2+D-14)y = 13e^{2x}$$

The auxiliary equation is $3m^2+m-14=0$

$$(x-2)(3x+7)=0$$

$3 \times 14 \times 3$

42 1

42

$-6\ 2$ 7

$\frac{-6}{3}$ $\frac{7}{3}$

$m = 2, -7/3 \quad C.F = Ae^{2x} + Be^{-7/3 x}$

$$P.I = \frac{1}{3D^2+D-14} 13e^{2x}$$

$$= \frac{1}{(3D+7)(D-2)} 13e^{2x}$$

$$= \frac{1}{(6+7)(D-2)} 13e^{2x}$$

$$= \frac{1}{\cancel{13}(D-2)} \cancel{13}e^{2x}$$

$$= \frac{1}{D-2} e^{2x} = \frac{1}{2-2} e^{2x} = \frac{1}{0} e^{2x} = xe^{2x}$$

$P.I = xe^{2x}$

$$\boxed{y = Ae^{2x} + Be^{-7/3 x} + xe^{2x}}$$

---

6. Solve $(D^2 - 2mD + m^2)y = e^{mx}$

Soln:-

Given that equation is $(D^2 - 2mD + m^2)y = e^{mx}$

The auxiliary equation is

m^2 $-2m$ $-m$ $-m$

$$(k^2 - 2mk + m^2) = 0$$

$$(k-m)(k-m) = 0$$

$$k = m, m$$

$$\therefore C.F = e^{mx}(Ax+B).$$

P.I is

$$P.I = \frac{1}{D^2 - 2mD + m^2} e^{mx}$$

$$= \frac{1}{(D-m)(D-m)} e^{mx}$$

$$P.I = \frac{x^2}{2} e^{mx}$$

The general equation is

$$y = C.F + P.I$$

$$\boxed{y = e^{mx}\left[Ax+B+x^2/2\right]}$$

7. Type : 6

Suppose $F(x) = x^m \sin ax$ (or) $x^m \cos ax$

Then the particular solution is

$$\frac{1}{\phi(D)}\left[x^m(\cos ax + i\sin ax)\right] = \frac{1}{\phi(D)} x^m e^{iax}$$

$$= e^{iax}\frac{1}{\phi(D)} x^m.$$

1. Solve : $(D^2+1)y = x^2\cos x$

soln :-

Given that equation is $(D^2+1)y = x^2\cos x$

The auxiliary equation is $m^2+1=0$

$$m = \pm i$$

$\therefore$ The complementary function is

Formula :-

$$e^{iax} = (\cos ax + i\sin ax).$$

$$C.F = A\cos x + B\sin x.$$

The particular equation is

$$P.I = \frac{1}{D^2+1} x^2 \cos x$$

$$= \text{Real Part of } \frac{1}{D^2+1} x^2 e^{ix}$$

$$= R.P. \text{ of } e^{ix} \frac{1}{(D+i)^2+1} x^2$$

$$= R.P. \text{ of } e^{ix} \frac{1}{D^2-1+2iD+1} x^2$$

$$= R.P. \text{ of } e^{ix} \frac{1}{D^2+2iD} x^2$$

$$= R.P. \text{ of } e^{ix} \frac{1}{2iD(1+D/2i)} x^2$$

$$= R.P. \text{ of } \frac{e^{ix}}{2iD}\left[1+D/2i\right]^{-1} x^2$$

$$= R.P. \text{ of } \frac{e^{ix}}{2iD}\left[1-D/2i+(D/2i)^2\right] x^2$$

$$= R.P. \text{ of } \frac{e^{ix}}{2iD} \times \frac{i}{i}\left[x^2-\frac{D(x^2)}{2i}-\frac{1}{4}D^2(x^2)\right]$$

$$= R.P. \text{ of } \frac{-ie^{ix}}{2D}\left[x^2-\frac{2x}{2i}-2/4\right]$$

$$= R.P. \text{ of } \frac{-ie^{ix}}{2D}\left[x^2-x/i-1/2\right]$$

$$= R.P. \text{ of } -\frac{e^{ix}}{2D}\left[ix^2-x-i/2\right]$$

$$= R.P. \text{ of } -\frac{e^{ix}}{2}\int\left[ix^2-x-i/2\right]dx$$

$$= R.P. \text{ of } (-1/2)e^{ix}\left[\frac{ix^3}{3}-\frac{x^2}{2}-\frac{ix}{2}\right]$$

$$= R.P. \text{ of } (-1/2)\left[\frac{ix^3}{3}-\frac{x^2}{2}-\frac{ix}{2}\right][\cos x+i\sin x]$$

$$= R.P.\ of\ (-1/2)\left[\frac{ix^3\cos x}{3} - \frac{x^3\cos x}{2} - \frac{ix\cos x}{2} - \frac{x^3\sin x}{3} - \frac{ix^2\sin x}{2} + \frac{x\sin x}{2}\right]$$

$$= (-1/2)\left[-\frac{x^2\cos x}{2} - \frac{x^3\sin x}{3} + \frac{x\sin x}{2}\right]$$

$$= [-1/2]\left[\frac{-x^2\cos x}{2} + \sin x\left[x/2 - x^3/3\right]\right]$$

$$P.I = \frac{x^2\cos x}{4} - \frac{1}{2}\sin x\,(x/2 - x^3/3)$$

$$\boxed{y = A\cos x + B\sin x + \frac{x^2\cos x}{4} - 1/2\,\sin x\,[x/2 - x^3/3]}$$

---

(2). Solve $(D^2-4D+3)y = \sin 3x \cos 2x$

Soln:-

Given that equation is

$(D^2-4D+3)y = \sin 3x \cos 2x$

The auxiliary equation is $m^2-4m+3=0$

3 ; -1, -1, -3

$(m-1)(m-3)=0$

$m = 1, 3.$

$C.F = Ae^x + Be^{3x}.$

$$P.I = \frac{1}{D^2-4D+3}\sin 3x\cos 2x$$

$$= \frac{1}{D^2-4D+3}\,\frac{1}{2}\left[\sin 5x + \sin x\right]$$

$$= \frac{1}{D^2-4D+3}\,\frac{1}{2}\sin 5x + \frac{1}{D^2-4D+3}\,1/2\sin x$$

$P.I = P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{-25-4D+3}\,\frac{1}{2}\sin 5x$$

$$= \frac{1}{-22-4D}\,\frac{1}{2}\sin 5x$$

$$= -\frac{1}{2}\,\frac{1}{4D+22}\sin 5x$$

$$= -\frac{1}{2}\left[\frac{4D-22}{(4D+22)(4D-22)}\right]\sin 5x$$

$$= -\frac{1}{2}\left[\frac{4D-22}{(16D^2)-(22)^2}\right]\sin 5x \qquad \begin{array}{r} 884\times 2 \\ \hline 1768 \\ \hline \end{array}$$

$$= -\frac{1}{2}\left[\frac{4D-22}{-400-484}\right]\sin 5x$$

$$= -\frac{1}{2}\left[\frac{4D-22}{-884}\right]\sin 5x = -\frac{1}{2}\left[\frac{4D-22}{-884}\right]\sin 5x$$

$$= \frac{1}{1768}\left[4D(\sin 5x) - 22\sin 5x\right]$$

$$= \frac{1}{1768}\left[4\cos 5x\,(5) - 22\sin 5x\right]$$

$$= \frac{1}{1768}\left(20\cos 5x - 22\sin 5x\right).$$

$$= \frac{1}{\cancel{1768}\;884}\,\cancel{2}\left(10\cos 5x - 11\sin 5x\right)$$

$$P.I_1 = \frac{1}{884}\left(10\cos 5x - 11\sin 5x\right).$$

$$P.I_2 = \frac{1}{-1-4D+3}\,\frac{1}{2}\sin x$$

$$= \frac{1}{2-4D}\,\frac{1}{2}\sin x$$

$$= \frac{2+4D}{(2-4D)(2+4D)}\,\frac{1}{2}\sin x$$

$$= \frac{2+4D}{4-16D^2}\,\frac{1}{2}\sin x$$

$$= -\frac{2+4D}{(16D^2-4)} \cdot \frac{1}{2}\sin x$$

$$= -\frac{2+4D}{-20} \cdot \frac{1}{2}\sin x = \frac{2+4D}{40}\sin x$$

$$= \frac{1}{40}\left[2\sin x + 4D\sin x\right]$$

$$= \frac{1}{40}\left[2\sin x + 4\cos x\right]$$

$$= \frac{1}{20}\left[\sin x + 2\cos x\right]$$

$$P.I = \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)$$

$$y = C.F + P.I$$

$$\boxed{y = Ae^{x} + Be^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)}$$

---

③. Solve : $(D^2+5D+6)y = e^x$

Soln :-

Given that equation is $(D^2+5D+6)y = e^x$

The auxiliary equation is $m^2+5m+6=0$

$$(m+2)(m+3)=0$$

$$m = -2, -3.$$

$$C.F = Ae^{-2x} + Be^{-3x}$$

$$P.I = \frac{1}{D^2+5D+6}e^x = \frac{1}{1+5+6}e^x$$

$$= \frac{1}{12}e^x$$

The general solution is

$$y = C.F + P.I$$

$$\boxed{y = Ae^{-2x} + Be^{-3x} + \frac{1}{12}e^x}$$

4. Solve: $(D^2+16)y = 2e^{-3x} + \cos 4x$

Soln:-

Given equation is $(D^2+16)y = 2e^{-3x} + \cos 4x$

The auxiliary equation is $m^2+16=0$

$$m^2 = -16$$

$$m = \pm 4i$$

$$C.F = A\cos 4x + B\sin 4x$$

$$P.I = \frac{1}{D^2+16}\left[2e^{-3x} + \cos 4x\right]$$

$$= \frac{1}{D^2+16} 2e^{-3x} + \frac{1}{D^2+16} \cos 4x$$

$$P.I_1 = \frac{1}{D^2+16} 2e^{-3x}$$

$$= 2\frac{1}{-9+16} e^{-3x} = \frac{2}{7} e^{-3x}$$

$$P.I_2 = \frac{1}{D^2+16} \cos 4x$$

$$= \frac{1}{-16+16} \cos 4x = x\frac{1}{2D} \cos 4x$$

$$= x\tfrac{1}{2}\int \cos 4x\, dx = x/2\left[\frac{\sin 4x}{4}\right]$$

$$= x/2 \frac{\sin 4x}{4}$$

$$= x/8 \sin 4x$$

$$P.I_2 = 1/8\, x \sin 4x$$

$$y = C.F + P.I$$

$$\boxed{y = A\cos 4x + B\sin 4x + 2/7\, e^{-3x} + 1/8\, x \sin 4x}$$

⑤. Solve: $(D^2-4D+3)y = e^{-x}\sin x$

Soln:-

Given that equation is

$$(D^2-4D+3)y = e^{-x}\sin x$$

The auxiliary equation is $m^2-4m+3=0$

$$(m-1)(m-3)=0$$

$$m=1,3.$$

$$C.F = Ae^{x}+Be^{3x}$$

$$P.I = \frac{1}{D^2-4D+3}e^{-x}\sin x$$

$$= e^{-x}\frac{1}{(D-1)^2-(D-1)4+3}\sin x$$

$$= e^{-x}\frac{1}{D^2+1-2D-4D+4+3}\sin x$$

$$= e^{-x}\frac{1}{D^2-6D+8}\sin x$$

$$= e^{-x}\frac{1}{-1-6D+8}\sin x = e^{-x}\frac{1}{7-6D}\sin x$$

$$= e^{-x}\frac{7+6D}{(7-6D)(7+6D)}\sin x$$

$$= e^{-x}\frac{7+6D}{49-36D^2}\sin x = e^{-x}\frac{7+6D}{49+36}\sin x$$

$$= e^{-x}\frac{7+6D}{85}\sin x = \frac{e^{-x}}{85}[7\sin x+6D\sin x]$$

$$P.I = \frac{e^{-x}}{85}[7\sin x+6\cos x]$$

$$\boxed{y = Ae^{x}+Be^{3x}+\frac{e^{-x}}{85}(7\sin x+6\cos x)}$$