19UMA06

DIFFERENTIAL EQUATIONS AND LAPLACE TRANSFORM

B.Sc. MATHEMATICS

III - SEMESTER

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UNIT-II

SECOND ORDER DIFFERENTIAL EQUATIONS WITH VARIABLE CO-EFFICIENTS, BOTH HOMOGENEOUS LINEAR EQUATIONS AND HOMOGENEOUS NON-LINEAR EQUATIONS.

Unit - II :-

Linear equations of second order with variable coefficient

Homogeneous equation of Euler type :-

Linear Differential Equations with variable coefficients.

- Linear equations of second order with variable coefficient
  - Homogeneous equation
    - Euler type (or) Cauchy - Euler type
    - Legendre's type
  - Non - homogeneous equation
    - No general method to solve

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x) \quad \to (1)$$

where $a_1, a_2, \dots a_n$ are constants and $f(x)$ is a function of x.

Equation (1) can be reduced to linear differential equation with constant co-efficients by putting the substitution.

$x = e^z$ (or) $z = \log x$.

Now, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$$x\frac{dy}{dx} = \frac{dy}{dz} \text{ (or) } x\frac{dy}{dx} = D'y \rightarrow ②$$

$$\left[\because \frac{dz}{dx} = \frac{d(\log x)}{dx} = 1/x\right]$$

where $D' = \frac{d}{dz}$

Note :-

The logarithm to the base e is known as the natural logarithm or the Napierian logarithm, after Napier, the inventor of logarithms.

In theoretical work, we use natural logarithms and so the suffix e is generally omitted, the base e being understood.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[1/x\frac{dy}{dz}\right]$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) \text{ [Apply (PRODUCT) Rule]}$$

$$= -1/x^2\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -1/x^2\frac{dy}{dz} + 1/x\frac{d^2y}{dz^2}(1/x)$$

$$\frac{d^2y}{dx^2} = -1/x^2\frac{dy}{dz} + 1/x^2\frac{d^2y}{dz^2}$$

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$= (D'^2 - D')y$$

$$x^2 \frac{d^2y}{dx^2} = D'(D'-1)y \rightarrow (3)$$

similarly,

$$x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y \rightarrow (4)$$

$$x^4 \frac{d^4y}{dx^4} = D'(D'-1)(D'-2)(D'-3)y \rightarrow (5)$$

and so on, Substituting (2), (3), (4), (5) and so on in (1), we get a differential equation with constant co efficients and can be solved by any one of the know methods.

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1). Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Soln:-

Given that equation is

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

$$[x^2D^2 - xD + 1]y = 0$$

Put $x = e^z$ $\qquad xD = D'$

$\log x = z$ $\qquad x^2D^2 = D'(D'-1)$

$$[D'(D'-1) - D' + 1]y = 0$$

$$[D'^2 - D' - D' + 1]y = 0$$

$$[D'^2 - 2D' + 1]y = 0$$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$$y = (Az + B)e^z$$

$$y = [A \log x + B]x$$

$$y = x[A \log x + B]$$

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2). Solve $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Soln:-

Given that equation is $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

$$[xD^2 + D]y = 0$$

Multiply by x

$$[x^2D^2 + xD]y = 0$$

$$[x^2D^2 + xD]y = 0$$

put $x = e^z$

$$\log x = z$$

$$xD = D'$$

$$x^2D^2 = D'(D'-1)$$

$$[D'(D'-1) + D']y = 0$$

$$[D'^2 - D' + D']y = 0$$

$$D'^2 y = 0$$

The auxiliary equation is $m^2=0$; $m=0,0$

$y=(Az+B)e^{0z}$ $\because$ since, $e^{0z}=1$.

$=Az+B$

$y=A(\log x)+B$

3). Solve $x^2y''+2xy'+2y=0$

Soln:-

Given that equation is $x^2y''+2xy'+2y=0$

$x^2D^2+2xD+2y=0$

$[x^2D^2+2xD+2]y=0 \longrightarrow (1)$

put $x=e^z$

$\log x=z$

$xD=D'$

$x^2D^2=D'(D'-1)$

$(1) \Rightarrow [D'(D'-1)+2D'+2]y=0$

$[D'^2-D'+2D'+2]y=0$

$[D'^2+D'+2]y=0$

The auxiliary equation is $m^2+m+2=0$

$m=\frac{-1\pm\sqrt{1-8}}{2}=\frac{-1\pm\sqrt{7}i}{2}$

$y=e^{-\frac{1}{2}z}\left[A\cos\frac{\sqrt{7}}{2}z+B\sin\frac{\sqrt{7}}{2}z\right]$

$=(e^z)^{-1/2}\left[A\cos\frac{\sqrt{7}}{2}z+B\sin\frac{\sqrt{7}}{2}z\right]$

$=x^{-1/2}\left[A\cos\left(\frac{\sqrt{7}}{2}\log x\right)+B\sin\left(\frac{\sqrt{7}}{2}\log x\right)\right]$

$$y = \frac{1}{\sqrt{x}}\left[A\cos\left(\frac{\sqrt{7}}{2}\log x\right) + B\sin\left(\frac{\sqrt{7}}{2}\log x\right)\right].$$

49. Solve $(x^2D^2 - 7xD + 12)y = x^2$

Solns:-

Given that equation is $(x^2D^2 - 7xD + 12)y = x^2$

$$\left[x^2\frac{d^2y}{dx^2} - 7x\frac{dx}{dy} + 12\right]y = x^2$$

i.e., $(x^2D^2 - 7xD + 12)y = x^2 \rightarrow (1)$

put $x = e^z$

$\log x = z$

$xD = D' \rightarrow (2)$

$x^2D^2 = D'(D'-1) \rightarrow (3)$

where $D' = \frac{d}{dz}$

$(1) \Rightarrow [D'(D'-1) - 7D' + 12]y = (e^z)^2$

$[D'^2 - 8D' + 12]y = e^{2z} \rightarrow (4)$

The auxiliary equation is $m^2 - 8m + 12 = 0$

$(m-6)(m-2) = 0$

$m = 2, 6$

$C.F = Ae^{2z} + Be^{6z}$

$= A(e^z)^2 + B(e^z)^6$

$C.F = Ax^2 + Bx^6$

$$P.I = \frac{1}{D'^2 - 8D' + 12}e^{2z}$$

$$= \frac{1}{4 - 16 + 12}e^{2z}$$

$$= z\frac{1}{2D' - 8}e^{2z}$$

$$= z\frac{1}{4-8}e^{2z}$$

$$= \frac{z}{-4}e^{2z}$$

$$= -\frac{z}{4}(e^{z})^2$$

$$= -\frac{z}{4}(e^{z})^2 = -\frac{\log x}{4}(x)^2$$

$$P.I = -\frac{x^2}{4}\log x$$

$\therefore y = C.F + P.I$

$$y = Ax^2 + Bx^6 - \frac{x^2}{4}\log x$$ //.

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5) Solve $(x^2D^2 + 4xD + 2)y = \log x$, given that when $x=1$, $y=0$ and $\frac{dy}{dx} = 0$.

Solns :-

Given that equation is

$$(x^2D^2 + 4xD + 2)y = \log x$$

Put $x = e^z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$$[D'(D'-1) + 4D' + 2]y = z$$

$$[D'^2 + 3D' + 2]y = z$$

The auxiliary equation is $m^2+3m+2=0$

$(m+1)(m+2)=0$

$m=-2,-1$

$C.F = Ae^{-1z}+Be^{-2z} = Ax^{-1}+Bx^{-2}$

$C.F = \frac{A}{x}+\frac{B}{x^2}$

$$P.I = \frac{1}{D'^2+3D'+2} z$$

$$= \frac{1}{2}\left[\frac{1}{1+\frac{D'^2+3D'}{2}}\right] z$$

$$= \frac{1}{2}\left[1+\frac{D'^2+3D'}{2}\right]^{-1} z$$

$$= \frac{1}{2}\left[1-\left(\frac{D'^2+3D'}{2}\right)+\left(\frac{D'^2+3D'}{2}\right)^2-\cdots\right] z$$

$$P.I = \frac{1}{2}\left[z-\frac{3}{2}\right] = \frac{1}{4}(2z-3) = \frac{1}{4}\left[2\log x-3\right]$$

$y = C.F + P.I$

$$y(x) = A\left(\frac{1}{x}\right)+B\left(\frac{1}{x^2}\right)+\frac{1}{4}\left[2\log x-3\right] \rightarrow (1)$$

(i) Given : $x=1 \Rightarrow y(1)=0$

$$\therefore y(x) = y(1) = A\left(\frac{1}{1}\right)+B\left(\frac{1}{1}\right)+\frac{1}{4}(2\log 1-3)$$

$$y(1) = A+B-\frac{3}{4}$$

$$0 = A+B-\frac{3}{4}$$

$$A+B = \frac{3}{4} \rightarrow (2).$$

(ii) Given : $x = 1 \Rightarrow y'(1) = 0$

$$y'(x) = A\left[\frac{-1}{x^2}\right] + B\left[\frac{-2}{x^3}\right] + \frac{1}{4}\left[\frac{2}{x} - 0\right]$$

$$= -\frac{A}{x^2} \cancel{+} - \frac{2B}{x^3} + \frac{1}{2x}$$

$$y'(1) = -A - 2B + \frac{1}{2}$$

$$A + 2B = \frac{1}{2} \longrightarrow (3)$$

$(2) - (3) \Rightarrow$

$$A + B = \frac{3}{4}$$
$$A + 2B = \frac{1}{2}$$
$$0 - B = \frac{1}{4}$$

$$\boxed{B = -\frac{1}{4}}$$

$B = -1/4$ is put in (2), we get

$(2) \Rightarrow \quad A - \frac{1}{4} = \frac{3}{4}$

$$A = \frac{3}{4} + \frac{1}{4}$$

$$\boxed{A = 1}$$

put in $A = 1$, $B = -1/4$ eqn (1), we get

(1) $\Rightarrow$

$$y = \frac{1}{x} - \frac{1}{4}\left(\frac{1}{x^2}\right) + \frac{1}{4}\left[2\log x - 3\right]$$

6). Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2 + \cos(\log x)$

Soln:-

Given that $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2 + \cos(\log x)$

Put $x = e^z$

$\log x = z$

$xD = D'$

$x^2 D^2 = D'(D'-1)$ where D' denotes $\frac{d}{dz}$

$[D'(D'-1) - 3D' + 4]y = (e^z)^2 + \cos z$

$[D'^2 - D' - 3D' + 4]y = e^{2z} + \cos z$

$[D'^2 - 4D' + 4]y = e^{2z} + \cos z$

The auxiliary equation is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$C.F = (Az + B)e^{2z} = (Az+B)(e^z)^2$

$= (A \log x + B)x^2$

$P.I_1 = \frac{1}{(D'-2)^2} e^{2z}$ Replace D' by a.

$= \frac{1}{(2-2)^2} e^{2z}$

$= z \frac{1}{2D'-4} e^{2z}$

$= z \frac{1}{4-4} e^{2z} = z^2 \frac{1}{2} e^{2z}$

$P.I_1 = \frac{z^2}{2}(e^z)^2 = \frac{(\log x)^2}{2} x^2 = \frac{x^2}{2}(\log x)^2$

$P.I_2 = \frac{1}{D'^2 - 4D' + 4} \cos z$

$$= \frac{1}{-1-4D'+4}\cos z$$ Replace D'^2 by $-a^2$

$$= \frac{1}{3-4D'}\cos z = \frac{3+4D'}{(3-4D')(3+4D')}\cos z$$

$$= \frac{3+4D'}{9-16D'^2}\cos z = \frac{1}{25}(3+4D')\cos z$$

$$= \frac{1}{25}\left[3\cos z + 4(-\sin z)\right]$$

$$= \frac{1}{25}\left[3\cos z - 4\sin z\right]$$

$$P.I_2 = \frac{1}{25}\left[3\cos(\log x) - 4\sin(\log x)\right]$$

$$y = C.F + P.I$$

$$y = (A\log x + B)x^2 + \frac{(\log x)^2}{2}x^2 + \frac{1}{25}\left[3\cos(\log x) - 4\sin(\log x)\right]$$

7). Solve $[x^2D^2 - xD + 1]y = \left[\frac{\log x}{x}\right]^2$

Soln:-

Given that $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$

put $x = e^z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$$\therefore [D'(D'-1) - D' + 1]y = \left(\frac{z}{e^z}\right)^2$$

$$[D'^2 - D' - D' + 1]y = z^2 e^{-2z}$$

$$[D'^2 - 2D' + 1]y = z^2 e^{-2z}$$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$C.F = (Az + B)e^z = [A \log x + B]x$$

$$P.I = \frac{1}{(D'-1)^2} z^2 e^{-2z}$$

$$= e^{-2z} \frac{1}{[(D'-2)-1]^2} z^2$$

$$= e^{-2z} \frac{1}{(D'-3)^2} z^2$$

$$= \frac{e^{-2z}}{3 \times 3} \left[\frac{1}{[1 - D'/3]^2} \right] z^2$$

$$= \frac{e^{-2z}}{9} \left[1 - D'/3 \right]^{-2} z^2$$

$$= \frac{e^{-2z}}{9} \left[1 + 2D'/3 + 3\left(\frac{D'}{3}\right)^2 + \dots \right] z^2$$

$$= \frac{e^{-2z}}{9} \left[z^2 + \frac{4z}{3} + \frac{2}{3} \right]$$

$$= \frac{(e^z)^{-2}}{9} \left[\frac{3z^2 + 4z + 2}{3} \right]$$

$$= \frac{(e^z)^{-2}}{27} \left[3z^2 + 4z + 2 \right]$$

$$= \frac{x^{-2}}{27} \left[3(\log x)^2 + 4(\log x) + 2 \right]$$

$$P.I = \frac{1}{27x^2}\left[3(\log x)^2 + 4\log x + 2\right]$$

$\therefore y = C.F + P.I$

$$y = (A\log x + B)x + \frac{1}{27x^2}\left[3(\log x)^2 + 4\log x + 2\right]$$

8). Solve $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$

Soln :-

Given that $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$

Put $x = e^z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$[D'(D'-1) - 2D' - 4]y = (e^z)^2 + 2z$

$[D'^2 - 3D' - 4]y = e^{2z} + 2z$

The auxiliary equation is $m^2 - 3m - 4 = 0$

$(m+1)(m-4) = 0$

$m = 4, -1$

$m = -1, 4$

$C.F = A(e^z)^{-1} + B(e^z)^4 = Ax^{-1} + Bx^4$

$C.F = A/x + Bx^4$

$$P.I_1 = \frac{1}{D'^2 - 3D' - 4}$$

$$= \frac{1}{4-6-4}e^{2z} = \frac{-1}{6}e^{2z} = -\frac{1}{6}(e^z)^2 = -\frac{1}{6}x^2$$

$$P.I_2 = \frac{1}{D'^2 - 3D' - 4} 2z = -\frac{1}{4 + 3D' - D'^2} 2z$$

$$= -\frac{1}{4}\left[\frac{1}{1+\left[\frac{3D'-D'^2}{4}\right]}\right] 2z$$

$$= -\frac{1}{4}\left[1 + \frac{3D'-D'^2}{4}\right]^{-1} 2z$$

$$= -\frac{1}{2}\left[1 - \left(\frac{3D'-D'^2}{4}\right)\right] z$$

$$= -\frac{1}{2}\left[z - \frac{3}{4}(1)\right] = -\frac{1}{2}z + \frac{3}{8} = -\frac{1}{2}\log x + \frac{3}{8}$$

$$y = C.F + P.I$$

$$y = \frac{A}{x} + Bx^4 - \frac{1}{6}x^2 - \frac{1}{2}\log x + 3/8$$

9). Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Soln :-

Given that equation is $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$$

$$x^2D^2 + xD = 12\log x \quad \rightarrow (1)$$

put $x = e^z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$① \Rightarrow [D'(D'-1) + D']y = 12z$

$[D'^2 - D' + D']y = 12z$

$D'^2 y = 12z$

The auxiliary equation is $m^2 = 0$

$m = 0, 0$

$C.F = (Az+B)e^{0z} = Az+B = A\log x + B$

$P.I = \frac{1}{(D'^2)} 12z = 12\frac{1}{(D'^2)} z$

$= 12\frac{1}{D'}\left[\frac{z^2}{2}\right] = 12\left[\frac{z^3}{6}\right]$

$= 2z^3$

$P.I = 2(\log x)^3$

$y = C.F + P.I$

$y = A\log x + B + 2(\log x)^3.$

10). Solve $(x^2D^2 + 4xD + 2)y = x\log x$

Soln:-

Given that $(x^2D^2 + 4xD + 2)y = x\log x \rightarrow ①$

put $x = e^z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$① \Rightarrow [D'(D'-1) + 4D' + 2]y = ze^z = e^z z.$

$[D'^2 + 3D' + 2]y = ze^z$

The auxiliary equation is $m^2+3m+2=0$

$$(m+1)(m+2)=0$$

$$m=-1,-2$$

$$C.F = Ae^{-2z}+Be^{-1z}$$

$$= A(e^{z})^{-2}+B(e^{z})^{-1}$$

$$C.F = Ax^{-2}+Bx^{-1} = A/x^2+B/x$$

$$P.I = \frac{1}{D'^2+3D'+2}\, ze^{z}$$

$$= e^{z}\frac{1}{D'^2+3D'+2}\, z$$

$$= e^{z}\frac{1}{(D'+1)^2+3(D'+1)+2}\, z$$

$$= e^{z}\frac{1}{D'^2+2D'+1+3D'+3+2}\, z$$

$$= e^{z}\left[\frac{1}{D'^2+5D'+6}\right] z = e^{z}\frac{1}{6}\left[\frac{1}{1+\frac{D'^2+5D'}{6}}\right] z$$

$$= e^{z}\frac{1}{6}\left[1+\left(\frac{D'^2+5D'}{6}\right)\right]^{-1} z = \frac{1}{6}e^{z}\left[1-\frac{D'^2+5D'}{6}+\dots\right] z$$

$$= \frac{1}{6}e^{z}\left[z-5/6\right] = \frac{1}{36}e^{z}(6z-5)$$

$$P.I = \frac{1}{36}x(6\log x-5).$$

$$y = C.F + P.I$$

$$y = \frac{A}{x^2}+\frac{B}{x}+\frac{1}{36}x(6\log x-5)$$

11). Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y = \sin(\log x)$.

Soln:

Given that $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y = \sin(\log x)$.

$x^2D^2+4xD+2y = \sin(\log x). \rightarrow ①$

put $x=e^z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$① \Rightarrow [D'(D'-1)+4D'+2]y = \sin z$

$[D'^2+3D'+2]y = \sin z \rightarrow ②$

The auxiliary equation is $m^2+3m+2=0$.

$(m+1)(m+2)=0$

$m=-1,-2$

C.F $=Ae^{-z}+Be^{-2z} = Ax^{-1}+Bx^{-2} = \frac{A}{x}+\frac{B}{x^2}$

P.I $= \frac{1}{D'^2+3D'+2}\sin z$

$= \frac{1}{-1+3D'+2}\sin z = \frac{1}{3D'+1}\sin z$

$= \frac{3D'-1}{(3D'+1)(3D'-1)}\sin z = \frac{3D'-1}{9D'^2-1}\sin z$

$= \frac{3D'-1}{-9-1}\sin z = \frac{3D'-1}{-10}\sin z$

$= -\frac{1}{10}[3D'\sin z - \sin z]$

$= -\frac{1}{10}[3\cos z - \sin z]$

P.I $= -\frac{3}{10}\cos z + \frac{1}{10}\sin z$

$$y = \frac{A}{x} + \frac{B}{x^2} - \frac{3}{10}\cos z + \frac{1}{10}\sin z$$

12). Solve : $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x + \pi$

Soln:-

Given that $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x + \pi$

$$x^2D^2 - xD + y = \log x + \pi \rightarrow ①$$

put

$x = e^z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$① \Rightarrow [D'(D'-1) - D' + 1]y = z + \pi$

$[D'^2 - 2D' + 1]y = z + \pi \rightarrow ②$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

$C.F = \cancel{Ae^z + Be^z}\ (Az+B)e^z = [A\log x + B]x$

$P.I = \frac{1}{(D'^2-1)^2}(z+\pi)$

$= (1-D'^2)^{-2}(z+\pi)$

$= [1 + 2D' + 3D'^2 + \dots](z+\pi)$

$P.I = z + \pi + 2 = \log x + \pi + 2$

$y = C.F + P.I$

$y = [A\log x + B]x + \log x + \pi + 2.$

13). Solve: $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \log \frac{\sin(\log x)+1}{x}$

Soln:-

Given that $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \log \frac{\sin(\log x)+1}{x}$

put $x = e^z$

$\log x = z$

$xD = D'$; $x^2D^2 = D'(D'-1)$

$$[D'(D'-1) - 3D' + 1]y = z\left[\frac{\sin z + 1}{e^z}\right]$$

$$[D'^2 - 4D' + 1]y = e^{-z} z(\sin z + 1)$$

$$= e^{-z} z \sin z + e^{-z} z$$

The auxiliary equation is $m^2 - 4m + 1 = 0$

$$m = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$m = 2 \pm \sqrt{3}$.

$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$\therefore$ C.F $= e^{2z}(A\cos\sqrt{3}z + B\sin\sqrt{3}z)$

$= (e^z)^2 (A\cos\sqrt{3}z + B\sin\sqrt{3}z)$

$\therefore$ C.F $= x^2[A\cos\sqrt{3}(\log x) + B\sin\sqrt{3}(\log x)]$

$$P.I_1 = \frac{1}{D'^2 - 4D' + 1} e^{-z} z$$

$$= e^{-z} \frac{1}{(D'^2-1) - 4(D'-1) + 1} z$$

$$= e^{-z} \frac{1}{D'^2 - 2D' + 1 - 4D' + 4 + 1} z = e^{-z}\frac{1}{D'^2 - 6D' + 6} z$$

$$= e^{-x}\frac{1}{6}\left[\frac{1}{1-D'+\frac{D'^2}{6}}\right]x$$

$$= e^{-x}\frac{1}{6}\left[1-D'+\frac{D'^2}{6}\right]^{-1} x = \frac{1}{6}e^{-x}\left[1-\left(D'-\frac{D'^2}{6}\right)\right]x$$

$$= \frac{e^{-x}}{6}\left[1+\left(D'-\frac{D'^2}{6}\right) - \left(\frac{D'-D'^2}{6}\right)^2+\cdots\right]x$$

$$= \frac{e^{-x}}{6}\left[x+1\right] = \frac{1}{6}\left[\frac{1}{x}\right](\log x+1)$$

$$P.I_2 = \frac{1}{D'^2-4D'+1}e^{-x}\,x\sin x$$

$$= e^{-x}\frac{1}{(D'^2-1)-4(D'-1)+1}x\sin x$$

$$= e^{-x}\frac{1}{D'^2-6D'+6}x\sin x$$

$$= e^{-x}\left[x\frac{1}{D'^2-6D'+6}\sin x - \frac{2D'-6}{[D'^2-6D'+6]^2}\sin x\right]$$

Formula $\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$

$$= e^{-x}\left[x\frac{1}{-1-6D'+6}\sin x - \frac{2D'-6}{[-1-6D'+6]^2}\sin x\right]$$

$$= e^{-x}\left[x\frac{1}{5-6D'}\sin x - \frac{2D'-6}{(5-6D')^2}\sin x\right]$$

$PI_2 = PI_{I} + PI_{II}$

$$PI_2 = xe^{-x}\frac{1}{5-6D'}\sin x - e^{-x}\frac{2D'-6}{(5-6D')^2}\sin x]$$

$$PI_I = x \frac{1}{5-6D'} \sin x$$

$$= x \frac{5+6D'}{(5-6D')(5+6D')} \sin x$$

$$= x \frac{5+6D'}{25-36D'^2} \sin x$$

$$= x \frac{5+6D'}{25+36} \sin x = x \frac{5+6D'}{61} \sin x$$

$$= \frac{x}{61} (5 \sin x + 6D' \sin x)$$

$$= \frac{x}{61} [5 \sin x + 6 \cos x]$$

$$PI_I = \frac{x}{61} [5 \sin x + 6 \cos x]$$

$$P.I_{II} = - \frac{2D'-6}{(5-6D')^2} \sin x$$

$$= - \frac{2D'-6}{25+36D'^2-60D'} \sin x$$

$$= - \frac{2D'-6}{25-36-60D'} \sin x$$

$$= - \frac{2D'-6}{-11-60D'} \sin x = \frac{2D'-6}{11+60D'} \sin x$$

$$= \frac{(2D'-6)(11-60D')}{121-3600D'^2} \sin x$$

$$= \frac{22D'-120D'^2-66+360D'}{121+3600} \sin x$$

$$= \frac{-120D'^2+382D'-66}{3721} \sin x$$

$$= \frac{1}{3721}\left[-120D'^2(\sin z) + 382D'(\sin z) - 66\sin z\right]$$

$$= \frac{1}{3721}\left[120\sin z + 382\cos z - 66\sin z\right] \quad \frac{\overset{\cancel{12}\cancel{0}}{120}}{\underline{54}}$$

$$= \frac{1}{3721}\left[54\sin z + 382\cos z\right]$$

$$PI_{II} = \frac{1}{3721}(54\sin z + 382\cos z)$$

$$= \frac{2}{3721}(27\sin z + 191\cos z)$$

$$PI_{II} = \frac{2}{3721}(27\sin z + 191\cos z)$$

$$P.I_2 = \frac{1}{e^z}\left[\frac{z}{61}[5\sin z + 6\cos z] + \frac{2}{3721}[27\sin z + 191\cos z]\right]$$

$$y = C.F + P.I$$

$$y = x^2\left[A\cos\sqrt{3}(\log x) + B\sin\sqrt{3}(\log x)\right]$$

$$+ \frac{1}{6}\left(\frac{1}{x}\right)[\log x + 1] + \frac{1}{x}\left[\frac{\log x}{61}[5\sin(\log x)\right.$$

$$\left. + 6\cos(\log x)] + \frac{2}{3721}[27\sin(\log x) + 191\cos(\log x)]\right]$$

$$\boxed{\begin{gathered} y = x^2\left[A\cos\sqrt{3}(\log x) + B\sin\sqrt{3}(\log x)\right] \\ + \frac{1}{6}\left(\frac{1}{x}\right)(\log x + 1) \\ + \frac{1}{x}\left[\left[\frac{\log x}{61}(5\sin(\log x + 6\cos(\log x)\right] + \frac{2}{3721}\left[27\sin(\log x) + 191\cos(\log x)\right]\right] \end{gathered}}$$

$_{W.}$

14. Solve :- $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left[x + \frac{1}{x}\right]$

Soln:-

Given that $[x^3D^3 + 2x^2D^2 + 2]y = 10\left[x + \frac{1}{x}\right]$

put $x = e^z$

$\log x = z$

So that, $xD = D'$; $x^2D^2 = D'(D'-1)$,

$x^3D^3 = D'(D'-1)(D'-2)$

$[D'(D'-1)(D'-2) + 2D'(D'-1) + 2]y = 10[e^z + e^{-z}]$

$[D'^3 - 3D'^2 + 2D' + 2D'^2 - 2D' + 2]y = 10[e^z + e^{-z}]$

$[D'^3 - D'^2 + 2]y = 10[e^z + e^{-z}]$

The auxiliary equation is $m^3 - m^2 + 2 = 0$

put $m = -1$ we get -

-1	1	-1	0	2
	0	-1	2	-2
	1	-2	2	0

$\therefore (m+1)(m^2 - 2m + 2) = 0$

(m_2, m_3). $m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$C.F = Ae^{m_1 z} + e^{\alpha z}[B\cos\beta z + C\sin\beta z]$

$= Ae^{-z} + e^z[B\cos z + C\sin z]$

$= A\frac{1}{x} + x[B\cos(\log x) + C\sin(\log x)]$

$C.F = A\frac{1}{x} + x[B\cos(\log x) + C\sin(\log x)]$.

$$P.I_1 = \frac{1}{D'^3 - D'^2 + 2} 10e^z$$

$$= 10 \frac{1}{1 - 1 + 2} e^z = 10 \frac{1}{2} e^z = 5e^z = 5x$$

$$P.I_2 = \frac{1}{D'^3 - D'^2 + 2} 10e^{-z}$$

$$= 10 \frac{1}{-1 - 1 + 2} e^{-z} = 10z \frac{1}{3D'^2 - 2D'} e^{-z}$$

$$= 10z \frac{1}{3+2} e^{-z} = 10z \frac{1}{5} e^{-z}$$

$$= 2ze^{-z}$$

$$= 2/x \log x$$

$$P.I_2 = \frac{2}{x} \log x$$

$$y = C.F + P.I$$

$$y = A\frac{1}{x} + x\left[B\cos(\log x) + C\sin(\log x)\right]$$

$$+ 5x + \frac{2}{x} \log x$$ //.

15. Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$.

soln:-

Given that $[x^2D^2 + xD + 1]y = \log x \sin(\log x)$.

put $x = e^z$

$\log x = z$

So, that, $xD = D'$; $x^2D^2 = D'(D'-1)$

$$[D'(D'-1) + D' + 1]y = z \sin z.$$

$$[D'^2 - D' + D' + 1]y = x \sin x$$

$$[D'^2 + 1]y = x \sin x$$

The auxiliary equation is $m^2 + 1 = 0$

$m = \pm i$

$C.F = A \cos x + B \sin x$

$C.F = A \cos(\log x) + B \sin(\log x)$

$$P.I = \frac{1}{D'^2 + 1}(x \sin x).$$

Formula $\frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$

$$P.I = x \cdot \frac{1}{D'^2+1} \sin x - \frac{2D'}{(D'^2+1)^2} \sin x$$

$$= x \cdot \frac{x}{2}(-\cos x) - \frac{2D'}{D'^2+1}\left[\frac{1}{D'^2+1} \sin x\right]$$

$$= -\frac{x^2}{2} \cos x - \frac{2D'}{D'^2+1}\left[-\frac{x}{2} \cos x\right]$$

$$= -\frac{x^2}{2} \cos x + \frac{1}{D'^2+1} D'(x \cos x)$$

$$= -\frac{x^2}{2} \cos x + \frac{1}{D'^2+1}[x(-\sin x) + \cos x]$$

$$= -\frac{x^2}{2} \cos x - \frac{1}{D'^2+1} x \sin x + \frac{1}{D'^2+1} \cos x$$

$$= -\frac{x^2}{2} \cos x - P.I + \frac{x}{2} \sin x$$

$$2P.I = -\frac{x^2}{2} \cos x + \frac{x}{2} \sin x$$

$\div 2$

$$P.I = -\frac{x^2}{4} \cos x + \frac{x}{4} \sin x.$$

$$P.I = -\frac{(\log x)^2[\cos(\log x)]}{4} + \frac{(\log x)\sin(\log x)}{4}$$

$y = C.F + P.I$

$$y = A\cos(\log x) + B\sin(\log x) - \frac{(\log x)^2[\cos(\log x)]}{4} + \frac{(\log x)[\sin(\log x)]}{}$$

16. Solve $x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3 \log x$

Soln:-

given that $[x^2D^2 + 2xD - 12]y = x^3 \log x$

put $x = e^z$

$\log x = z$

So that, $xD = D'$, $x^2D^2 = D'(D'-1)$

$[D'(D'-1) + 2D' - 12]y = (e^z)^3 z$

$[D'^2 - D' + 2D' + 12]y = ze^{3z}$

$[D'^2 + D' + 12]y = ze^{3z}$

The auxiliary equation is $m^2 + m + 12 = 0$

$(m+4)(m-3) = 0$

$m = 3, -4.$

$\therefore C.F = Ae^{-4z} + Be^{3z}$

$\therefore C.F = A(e^z)^{-4} + B(e^z)^3 = Ax^{-4} + Bx^3$

$$P.I = \frac{1}{D'^2 + D' - 12} ze^{3z}$$

$$= e^{3z}\frac{1}{(D'+3)^2 + (D'+3) - 12} z$$

$$= e^{3z} \frac{1}{D'^2 + 6D' + 9 + D' + 3 - 12} z$$

$$= e^{3z} \frac{1}{D'^2 + 7D'} z$$

$$= e^{3z} \frac{1}{7D'\left[1 + \frac{D'}{7}\right]} z$$

$$= e^{3z} \frac{1}{7D'}\left[1 + \frac{D'}{7}\right]^{-1} z$$

$$= e^{3z} \frac{1}{7D'}\left[1 - \frac{D'}{7} + \cdots\right] z$$

$$= e^{3z} \frac{1}{7D'}\left[z - \frac{1}{7}\right]$$

$$= e^{3z}\left[\frac{z^2}{2} - \frac{1}{7} z\right] = \frac{e^{3z}}{98}\left[7z^2 - 2z\right]$$

$$= \frac{(e^z)^3}{98} z\left[7z - 2\right]$$

$$P.I = \frac{x^3}{98} \log x\left[7\log x - 2\right]$$

$$\therefore\ y = C.F + P.I$$

$$= Ae^{-4z} + Be^{3z} + \frac{x^3}{98} \log x\left[7\log x - 2\right]$$

17. Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Soln :- Given that $[x^2D^2 - 3xD + 4]y = (1+x)^2$ -

put $x = e^z$

$\log x = z$

So, that , $xD = D'$; $x^2D^2 = D'(D'-1)$

$(1) \Rightarrow [D'(D'-1) - 3D' + 4]y = [1+e^z]^2$

$[D'^2 - D' - 3D' + 4]y = [1+e^z]^2$

$[D'^2 - 4D' + 4]y = 1 + (e^z)^2 + 2e^z$

$[D' - 4D' + 4]y = e^{0z} + e^{2z} + 2e^z$

$[D' - 2]^2 y = e^{0z} + e^{2z} + 2e^z$

The auxiliary equation is $(m-2)^2 = 0$

$m = 2, 2$

$C.F = (Az + B)e^{2z}$

$C.F = (A \log x + B)x^2$

$P.I_1 = \frac{1}{(D'-2)^2} e^{0z} = \frac{1}{4} e^{0z} = \frac{1}{4}$

$P.I_2 = \frac{1}{(D'-2)^2} e^{2z} = \frac{1}{(2-2)^2} e^{2z} = z \frac{1}{2(D'-2)} e^{2z}$

$= \frac{z}{2} \frac{1}{2-2} e^{2z} = \frac{z^2}{2} \cancel{\frac{1}{2}} e^{2z}$

$P.I_2 = \frac{z^2}{2} e^{2z} = \frac{(\log x)^2}{2} x^2$

$P.I_3 = \frac{1}{(D'-2)^2} 2e^z = 2 \frac{1}{(1-2)^2} e^z$

$P.I_3 = 2e^z = 2x$

$y = C.F + P.I$

$y = (A \log x + B)x^2 + \frac{1}{4} + \frac{(\log x)^2}{2} x^2 + 2x$

18. Solve $x \frac{d^2y}{dx^2} - \frac{2y}{x} = x + \frac{1}{x^2}$

Soln:-

Given that $x^2 \frac{d^2y}{dx^2} - 2y = x\left[x + \frac{1}{x^2}\right]$

$[x^2 D^2 - 2]y = x^2 + \frac{1}{x} \rightarrow (1)$

put, $x = e^z$

$\log x = z$

So that, $xD = D'$

$x^2D^2 = D'(D'-1)$

(1) $\Rightarrow [D'(D'-1)-2]y = (e^z)^2 + \frac{1}{e^z}$

$[D'^2 - D' - 2]y = e^{2z} + e^{-z}$

The auxiliary equation is $m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2, -1$

$\therefore CF = Ae^{-z} + Be^{2z} = Ax^{-1} + Bx^2$

$$P.I_1 = \frac{1}{D'^2 - D' - 2} e^{2z}$$

$$= \frac{1}{4-2-2} e^{2z} = z \frac{1}{2D'-1} e^{2z}$$

$$P.I_1 = z \frac{1}{4-1} e^{2z} = z \frac{1}{3} e^{2z} = \frac{\log x}{3} x^2$$

$$P.I_2 = \frac{1}{D'^2 - D' - 2} e^{-z}$$

$$= \frac{1}{1+1-2} e^{-z} = z \frac{1}{2D'-1} e^{-z}$$

$$= z \frac{1}{-2-1} e^{-z} = \frac{z}{-3} e^{-z}$$

$$P.I_2 = -\frac{\log x}{3} x^{-1} = \frac{-1}{3x}(\log x)$$

$y = C.F + P.I$

$$y = A\cancel{x}_x + Bx^2 + \frac{x^3}{3}\log x - \frac{1}{3x}\log x$$

$$y = \frac{A}{x} + Bx^2 + \frac{x^3}{3}\log x - \frac{1}{3x}\log x$$

19. Find the equation of the curve which statistics the differential equation $4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + y = 0$ and touches x-axis at an angle of 60° at $x = 1$.

Soln:-

$$\text{Let } x = e^z$$

$$\log x = z$$

$$x^2\frac{d^2y}{dx^2} = D'(D'-1)y \quad ; \quad x\frac{dy}{dx} = D'y$$

$$[4D'(D'-1) - 4D' + 1]y = 0$$

$$[4D'^2 - 4D' - 4D' + 1]y = 0$$

$$[4D'^2 - 8D' + 1]y = 0$$

The auxiliary equation is $4m^2 - 8m + 1 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{8 \pm \sqrt{64 - 16}}{8} = \frac{8 \pm \sqrt{48}}{8} = \frac{8 \pm 4\sqrt{3}}{8}$$

$$m = 1 \pm \frac{\sqrt{3}}{2}$$

$$C.F = Ae^{(1+\frac{\sqrt{3}}{2})z} + Be^{(1-\frac{\sqrt{3}}{2})z}$$

$$y = Ax^{(1+\frac{\sqrt{3}}{2})} + Bx^{(1-\sqrt{3}/2)z} \rightarrow (1)$$

Given $y = 0$ when $x = 1$

$$0 = A + B \rightarrow (2)$$

$$\frac{dy}{dx} = \tan 60^\circ \text{ at } x = 1$$

$$\frac{dy}{dx} = \sqrt{3} \text{ at } x = 1$$

Differenciate with respect to x in (1).

$$\frac{dy}{dx} = A\left(1+\frac{\sqrt{3}}{2}\right)x^{1+\frac{\sqrt{3}}{2}-1} + B(1-\sqrt{3}/2)x^{1-\frac{\sqrt{3}}{2}-1}$$

4 -8

$\cancel{y'} = A\left[1+\frac{\sqrt{3}}{2}\right] x^{\sqrt{3}/2} + B\left(1-\sqrt{3}/2\right) x^{-\sqrt{3}/2}$

$\sqrt{3} = A(1+\sqrt{3}/2) + B(1-\sqrt{3}/2)$

$\sqrt{3} = A + A\sqrt{3}/2 + B - B\sqrt{3}/2$

Sub in (2)

$\sqrt{3} = \frac{\sqrt{3}}{2} A - \sqrt{3}/2 B = \sqrt{3}/2 (A-B)$

$1 = 1/2 (A-B)$

$A - B = 2 \rightarrow (3).$

Adding (2) and (3)

$2A = 2$

$A = 1$

Sub $A = 1$ in (2)

$B = -1$

eqn (1) becomes

$y = x^{(1+\sqrt{3}/2)} - x^{(1-\sqrt{3}/2)}$

20. The Radial displacement u in a rotating disc at a distance r from the axis is given by $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + kr = 0$ find its displacement.

Soln:-

Given that $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + kr = 0$

$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -kr$

Multiple by r^2 on bothsides,

$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -kr^3$

Let $r = e^z$

$z = \log r$

$$r^2 \frac{d^2u}{dr^2} = D'(D'-1)u \quad ; \quad r\frac{du}{dr} = D'u$$

$$[D'(D'-1) + D' - 1]u = -kr^3$$

$$[D'^2 - D' + D' - 1]u = -ke^{3z}$$

$$[D' - 1]u = -kr^3$$

The auxiliary equation is $m^2 - 1 = 0$

$$m = \pm 1 \quad ; \quad m = 1, -1$$

$$C.F = Ae^{z} + Be^{-z}$$

$$P.I = \frac{1}{D'-1}(-ke^{3z})$$

$$= \frac{1}{9-1}(-ke^{3z})$$

$$P.I = -\frac{1}{8}ke^{3z}$$

$$U = Ae^{z} + Be^{-z} - \frac{1}{8}ke^{3z}$$

$$U = Ar + Br^{-1} - \frac{k}{8}r^3$$

$$U = Ar + B/r - k/8\, r^3$$

21. Reduce $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = x$ into a differencial equation with constant coefficients

Soln:

Given that $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = x$

$$[x^2D^2 - 3xD + 3]y = x \quad \rightarrow (1)$$

Put $x = e^z$; $\log x = z$.

$xD = D'$, $x^2D^2 = D'(D'-1)$

$(1) \Rightarrow [D'(D'-1) - 3D' + 3]\, y = e^z$

$$[D'^2 - 4D' + 3]\, y = e^z$$

22. Write Euler's Homogeneous linear differential equation. How will you convert it to a linear differential equation with constant coefficients?

Soln:-

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x) \rightarrow (1)$$

Where $a_0, a_1, a_2, \dots, a_n$ are constants and $f(x)$ is a function of x.

Equation (1) can be reduced to linear differential equation with constant coefficients by putting the substitution.

$x = e^z$ (or) $z = \log x$

$xD = D'$, $D' = \frac{d}{dz}$

$x^2D^2 = D'(D'-1)$

$x^3D^3 = D'(D'-1)(D'-2)$

23. Transform $(x^2D^2 + xD + 1)y = 0$ into differential equations with constant coefficients, where $D = \frac{d}{dx}$.

Soln:-

Given that $(x^2D^2 + xD + 1)y = 0 \rightarrow (1)$

put $x = e^z$

$\log x = z$

$xD = D'$; $x^2D^2 = D'(D'-1)$

(1) => $[D'(D'-1) + D' + 1]y = 0$

$[D'^2 - D' + D' + 1]y = 0$

$[D'^2 + 1]y = 0$

PROBLEMS BASED ON LEGENDRE'S LINEAR DIFFERENTIAL EQUATION

[EQUATION REDUCIBLE TO LINEAR FORM]

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = 0 \quad \text{---}(1)$$

Where k's are constants and Q is a function of x is called Legendre's linear differential equations. Such equations canbe reduced to linear equations with constant coefficients by putting

$ax+b = e^z$

$z = \log(ax+b)$

If $D' = \frac{d}{dz}$, then $(ax+b)D = aD'$

$(ax+b)^2 D^2 = a^2 D'(D'-1)$

and so on.

After making these subsitutions in (1), it reduces to a linear differential equation with constant coefficients.

Legendre's linear differential equation canbe reduced to the Euler homogeneous linear forms also, by putting $ax+b=t$ and solved by the Euler's method.

24. Solve $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4$

Soln:-

Given that

$$\left[(x+2)^2 D^2 - (x+2)D + 1\right]y = 3x+4 \quad \rightarrow (1)$$

put $x+2 = e^t \Rightarrow x = e^t - 2$

$$\log(x+2) = t$$

$$(x+2)D = D'$$

$$(x+2)^2 D^2 = D'(D'-1)$$

$$(1) \Rightarrow [D'(D'-1) - D' + 1]y = 3[e^t - 2] + 4$$

$$[D'^2 - D' - D' + 1]y = 3e^t - 6 + 4$$

$$[D'^2 - 2D' + 1]y = 3e^t - 2$$

The auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$$C.F = (At+B)e^t = [A \log(x+2) + B](x+2)$$

$$P.I_1 = \frac{1}{(D'-1)^2} 3e^t$$

$$= 3\frac{1}{(1-1)^2} e^t$$ [Replace D' by 1, ordinary rule fails]

$$= t \frac{1}{2(D'-1)} 3e^t$$

$$= \frac{t}{2} \frac{1}{1-1} 3e^t = \frac{t^2}{2} 3e^t = \frac{3}{2} t^2 e^t$$

$$P.I_1 = \frac{3}{2}\left[\log(x+2)\right]^2 (x+2)$$

$$P.I_2 = \frac{1}{(D'-1)^2}[-2e^{0z}] = -2\frac{1}{(-1)^2} = -2$$

$$y = C.F + P.I$$

$$y = [A\log(x+2)+B](x+2) + \frac{9}{2}[\log(x+2)]^2(x+2)$$

$$-2 \quad z_4$$

25. $[(x+1)^2D^2 + (x+1)D + 1]y = A\cos[\log(x+1)]$

solve.

soln:-

Given that $[(x+1)^2D^2 + (x+1)D+1]y = 4\cos[\log(x+1)]$ $\rightarrow (1)$

put $1+x = e^z$

$z = \log(1+x)$

$(x+1)D = D'$; $(x+1)^2D^2 = D'(D'-1)$

$x = e^z - 1$

(1) $\Rightarrow$ $[D'(D'-1)+D'+1]y = 4[\cos z]$

$[D'^2 - D' + D' + 1]y = A[\cos z]$

$[D'^2+1]y = 4\cos z$

The auxiliary equation is $m^2+1=0$

$m = \pm i$

$C.F = [A\cos z + B\sin z]$

$C.F = A\cos[\log(x+1)] + B\sin[\log(x+1)]$

$P.I = \frac{1}{D'^2+1}4\cos z = 4\frac{1}{-1+1}\cos z$ Replace $[D'^2$ by -1^2

$= 4\frac{1}{0}\cos z = 4z\frac{1}{2D'}\cos z$ ordinary rule fails$]$

$= 2z\frac{1}{D'}\cos z = 2z\int\cos z\,dz = 2z\sin z$

$$P.I = 2\log(x+1)\ \sin[\log(1+x)]$$

$$\therefore y = C.F + P.I$$

$$y = A\cos[\log(x+1)] + B\sin[\log(x+1)]$$
$$+ 2\log(x+1)\sin[\log(x+1)]$$

26. Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

Soln:-

Given that

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$$

$$[(1+x)^2 D^2 + (1+x)D + 1]y = 2\sin[\log(1+x)] \rightarrow (1)$$

put $1+x = e^z$

$$z = \log(1+x)$$

$$(1+x)D = D' \ ; \ (x+1)^2 D^2 = D'(D'-1)$$

$$(1) \Rightarrow [D'(D'-1) + D' + 1]y = 2\sin z$$

$$[D'^2 - D' + D' + 1]y = 2\sin z$$

$$[D'^2 + 1]y = 2\sin z$$

The auxiliary equation is $m^2+1=0$

$$m = \pm i$$

$$C.F = A\cos z + B\sin z = A\cos[\log(1+x)] + B\sin[\log(1+x)]$$

$$P.I = \frac{1}{D'^2+1} 2\sin z$$

$$= 2\frac{1}{-1+1}\sin z = 2\frac{1}{0}\sin z$$

$$= 2z\frac{1}{2D'}\sin z = z\frac{1}{D'}\sin z$$

$$= z\int \sin z\, dz = z[-\cos z] = -z\cos z.$$

$$P.I = -\log(1+x)\cos[\log(1+x)]$$

$\therefore y = C.F + P.I$

$$y = A\cos[\log(1+x)] + B\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$$ /4.

27. Solve $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2+4x+1$

Soln:- Given that

$$[(3x+2)^2D^2 + 3(3x+2)D - 36]y = 3x^2+4x+1 \rightarrow (1)$$

put $3x+2 = e^z$

$\log(3x+2) = z$

$3x = e^z - 2$

$x = \dfrac{e^z}{3} - 2/3$

Let $(3x+2)D = 3D'$

$(3x+2)^2D^2 = 9D'(D'-1)$

$(1) \Rightarrow [9D'(D'-1) + 3(3D') - 36]y = 3\left[\frac{1}{3}e^z - \frac{2}{3}\right]^2 + 4\left[\frac{1}{3}e^z - \frac{2}{3}\right] + 1.$

$[9D'^2 - 9D' + 9D' - 36]y = 3\left[\frac{1}{9}e^{2z} + \frac{4}{9} - \frac{4}{9}e^z\right] + \frac{4}{3}e^z - \frac{8}{3} + 1$

$[9D'^2 - 36]y = 1/3\, e^{2z} + \frac{4}{3} - \frac{4}{3}e^z + \frac{4}{3}e^z - 8/3 + 1$

$= 1/3\, e^{2z} - 4/3 + 1 = \frac{1}{3}e^{2z} - \frac{1}{3}$

$\div 9 \Rightarrow [D'^2 - 4]y = \frac{1}{27}e^{2z} - \frac{1}{27}$

The auxiliary equation is $m^2 - 4 = 0$

$m^2 = 4$

$m = -2, 2 \Rightarrow m = \pm 2.$

$C.F = Ae^{2z} + Be^{-2z} = A(e^z)^2 + B(e^z)^{-2}$

$C.F = A(3x+2)^2 + B(3x+2)^{-2}$ /1.

$$P.I_1 = \frac{1}{D'^2-4}\,\frac{e^{2z}}{27}$$

$$= \frac{1}{27}\,\frac{1}{4-4}\,e^{2z}$$ (Replace D^2 by 2 Ordinary rule fails)

$$= \frac{1}{27}\,z\,\frac{1}{2D'}\,e^{2z}$$

$$= \frac{1}{54}\,z\,\frac{1}{2D'}\,e^{2z} = \frac{1}{54}\,z\,\frac{e^{2z}}{2} = \frac{z}{108}\,e^{2z}$$

$$P.I_1 = \frac{z}{108}(e^z)^2 = \frac{\log(3x+2)}{108}(3x+2)^2$$

$$P.I_2 = \frac{1}{D'^2-4}\,\frac{e^{0z}}{27}$$

$$= \left(\frac{1}{27}\right)\left(\frac{1}{-4}\right)e^{0z} = -\frac{1}{108}$$

$\therefore y = C.F + P.I$

$$y = A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}(3x+2)^2\log(3x+2) + \frac{1}{108}$$ //.

28. Solve $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

Soln:-

Given that $[(2x+3)^2D^2 + (2x+3)D - 12]\,y = 6x$ → (1)

put $(2x+3) = e^z$

$$\log(2x+3) = z$$

$$2x = e^z - 3$$

$$x = \frac{e^z}{2} - 3/2$$

$$(2x+3)D = 2D' \;;\; (2x+3)^2D^2 = 4D'(D'-1)$$

$$\therefore (1) \Rightarrow [4D'(D'-1) - 2D' - 12]\,y = 6\left[\frac{e^z}{2} - 3/2\right]$$

$$[4D'^2 - 4D' - 2D' - 12]\,y = \frac{6}{2}[e^z - 3]$$

$[4D'^2 - 6D' - 12]y = 3e^z - 9$

$\div 4$ $[D'^2 - \frac{3}{2}D' - 3]y = \frac{1}{4}[3e^z - 9]$

The auxiliary equation is $m^2 - \frac{3}{2}m - 3 = 0$

$$m = \frac{\frac{3}{2} \pm \sqrt{9/4 + 12}}{2} = \frac{\frac{3}{2} \pm \sqrt{57/4}}{2} = \frac{3 \pm \sqrt{57}}{4}$$

$\therefore$ Let $a = \dfrac{3+\sqrt{57}}{4}$; $b = \dfrac{3-\sqrt{57}}{4}$

$$\therefore C.F = Ae^{az} + Be^{bz}$$

$$= A(e^z)^a + B(e^z)^b$$

$$C.F = A(2x+3)^a + B(2x+3)^b$$

$$P.I_1 = \frac{1}{D'^2 - \frac{3}{2}D' - 3}\left(\frac{3e^z}{4}\right)$$

$$= \frac{3}{4}\,\frac{1}{1 - 3/2 - 3}\,e^z \quad \text{Replace } D' \text{ by } 1$$

$$= \frac{3}{4}\,\frac{1}{-7/2}\,e^z = \frac{3}{4}\left(\frac{-2}{7}\right)e^z = \frac{-3}{14}e^z$$

$$P.I_1 = -3/14\,(2x+3)$$

$$P.I_2 = \frac{1}{D'^2 - \frac{3}{2}D' - 3}\left(-9/4\,e^{0z}\right) = \frac{-9}{4}\left(\frac{1}{-3}\right) = \frac{3}{4}$$

$$\therefore y = C.F + P.I_1 + P.I_2$$

$$y = A(2x+3)^a + B(2x+3)^b - \frac{3}{14}(2x+3) + \frac{3}{4}$$ //.

Where $a = \dfrac{3+\sqrt{57}}{4}$; $b = \dfrac{3-\sqrt{57}}{4}$ //.

29. Transform the equation

$$(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$$ into a linear differential equation with constant coefficients.

Soln:- Given that

$$[(2x+3)^2 D^2 - 2(2x+3)D - 12]\, y = 6x \rightarrow (1)$$

put $2x+3 = e^z$

$\log(2x+3) = z$

$2x = e^z - 3$

$x = \frac{e^z}{2} - \frac{3}{2}$

Let $(2x+3)D = 2D'$; $(2x+3)^2 D^2 = 4D'(D'-1)$

$\therefore (1) \Rightarrow [4D'(D'-1) - 4D' - 12]\, y = 6\left[\frac{e^z}{2} - \frac{3}{2}\right]$

$[4D'^2 - 4D' - 4D' - 12]\, y = \frac{6}{2}[e^z - 3]$

$[4D'^2 - 8D' - 12]\, y = 3e^z - 9$

$\div 4$, $[D'^2 - 2D' - 3]\, y = \frac{1}{4}[3e^z - 9]$

The auxiliary equation is $m^2 - 2m - 3 = 0$

$(m+1)(m-3) = 0$

$m = -1, 3$

$C.F = Ae^{-z} + Be^{3z} = A(e^z)^{-1} + B(e^z)^3$

$C.F = A(2x+3)^{-1} + B(2x+3)^3$.

$P.I_1 = \frac{1}{D'^2 - 2D' - 3} \frac{1}{4} 3e^z$

$$= \frac{3}{4} \frac{1}{1-2-3} e^{z} \quad \text{Replace } D' \text{ by } 1$$

$$= \frac{3}{4} \frac{1}{-4} e^{z}$$

$$P.I_1 = -\frac{3}{16} e^{z} = -\frac{3}{16}(2x+3).$$

$$P.I_2 = \frac{1}{D'^2 - 2D' - 3} - \frac{9}{4} e^{0z}$$

$$= \frac{1}{-3}\left(-\frac{9}{4}\right)$$

$$P.I_2 = \frac{9}{12} = \frac{3}{4}$$

$$y = C.F + P.I$$

$$\boxed{y = A(2x+3) + B(2x+3)^3 - \frac{3}{16}(2x+3) + \frac{3}{4}}$$