19UMA02

DIFFERENTIAL CALCULUS

B.Sc. MATHEMATICS

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UNIT-I

PARTIAL DERIVATIVES,
HIGHER DERIVATIVES,
HOMOGENEOUS FUNCTION,
TOTAL DIFFERENTIAL CO EFFICIENT,
IMPLICIT FUNCTION,
PROBLEMS.

DIFFERENTIAL CALCULUS

UNIT - I

Partial derivative

Let $f(x,y)$ be a function of two variables x and y. Suppose x vary while keeping y fixed say $y=k$, where k is a constant. Then a function of single variable x, namely $g(x) = f(x,k)$.

If g has a derivative at h, then the partial derivative of f with respect to x at (h,k) and denote it by $\frac{\partial f}{\partial x}$ of f_x.
Similarly, we can define the partial derivative of $f(x,y)$ with respect to y.

Thus we have

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

and

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{k \to 0} \frac{f(x_0;y_0+k) - f(x_0,y_0)}{k}$$

If the point (x_0,y_0) vary f_x and f_y become functions of two variables defined by

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

Higher derivatives

If f is a function of two variables then its partial derivatives f_x and f_y are also functions of two variables. So that we consider their partial derivatives $(f_x)_x$, $(f_y)_x$ and $(f_y)_y$ which are called second order partial derivatives.

If $z = f(x,y)$, then

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus, the notation f_{xy} or $\frac{\partial^2 f}{\partial y \partial x}$ means we first differentiate f with respect to x and then with respect to y.

Homogeneous function

A function $f(x,y)$ of two independent variables x and y is said to be homogeneous in x and y of degree n if $f(tx, ty) = t^n f(x,y)$ for any positive quantity t where t is independent of x and y.

Example

Suppose $f(x,y) = \dfrac{x^3+y^3}{x+y}$

$$f(tx,ty) = \frac{t^3x^3+t^3y^3}{tx+ty} = \frac{t^{\cancel{3}\,2}(x^3+y^3)}{\cancel{t}(x+y)} = t^2 f(x,y).$$

$\therefore$ $f(x,y)$ is a homogeneous function of degree 2 in x and y.

Note :-

If $f(x,y)$ is a homogeneous function of degree n in x and y, then we can express $f(x,y)$ as $f(x,y) = x^n \phi(y/x)$.

Euler's Theorem on homogeneous functions

If f is a homogenous function of degree n in x and y. then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$.

Proof

Since f is a homogeneous function of degree n in x and y. we can write f as

$$f = x^n F(y/x) \longrightarrow (1)$$

Differentiating (1) partially w.r.to x,

$$\frac{\partial f}{\partial x} = x^n F'(y/x)(-y/x^2) + nx^{n-1}F(y/x)$$

$$x\frac{\partial f}{\partial x} = -x^{n-1}y\,F'(y/x) + nx^n F(y/x) \longrightarrow (2)$$

Differentiating (1) partially w.r.to y.

$$\frac{\partial f}{\partial y} = x^n F'(y/x)(1/x)$$

$$y\frac{\partial f}{\partial y} = y\,x^{n-1}F'(y/x) \longrightarrow (3).$$

Adding (2) and (3).

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nx^n F(y/x)$$

i.e, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf.$

Note:-

The generalised form of Euler's theorem for homogeneous function of degree n in k variables $x_1, x_2, \ldots, x_k$ is

$$x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+x_3\frac{\partial f}{\partial x_3}+\ldots+x_n\frac{\partial f}{\partial x_n}=kf;$$

Problems:-

① If $f(x,y)=\log\sqrt{x^2+y^2}$ show that $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$.

Solution:

$$f(x,y)=\log\sqrt{x^2+y^2}$$
$$=\log(x^2+y^2)^{1/2}$$
$$f(x,y)=\tfrac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial f}{\partial x}=\frac{1}{\cancel{2}}\frac{1}{(x^2+y^2)}\cancel{2}x=\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}$$

$$=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$=\frac{y^2-x^2}{(x^2+y^2)^2}\longrightarrow ①.$$

$$\frac{\partial f}{\partial y} = \frac{1}{\cancel{2}}\frac{1}{x^2+y^2}\cancel{2}y = y/x^2+y^2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2} \longrightarrow (2).$$

Adding (1) and (2), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$= \frac{\cancel{y^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2}$$

$$\boxed{\text{i.e. } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

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(8). If $z = e^x(x\cos y - y\sin y)$. Show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Solution:-

$$z = e^x(x\cos y - y\sin y).$$

$$\frac{\partial z}{\partial x} = e^x[(x\cdot 0 + \cos y\cdot 1) - 0] + e^x(x\cos y - y\sin y)$$

$$= e^x\cos y + e^x(x\cos y - y\sin y)$$

$$\frac{\partial^2 z}{\partial x^2} = e^x\cdot 0 + e^x\cos y + e^x[x\cdot 0 + \cos y\cdot 1 - 0] + e^x(x\cos y - y\sin y).$$

$$= e^x\cos y + e^x\cos y + e^x(x\cos y - y\sin y)$$

$$= 2e^x\cos y + e^x(x\cos y - y\sin y)$$

$$= e^x(2\cos y + x\cos y - y\sin y) \longrightarrow (1).$$

$$\frac{\partial z}{\partial y} = e^x [(x(-\sin y) + \cos y \cdot 0) - (y\cos y + \sin y \cdot 1)] + 0 \, (x\cos y - y\sin y).$$

$$= e^x[-x\sin y - y\cos y - \sin y]$$

$$\frac{\partial^2 z}{\partial y^2} = e^x [-(x\cos y + \sin y \cdot 0) - (y(-\sin y) + \cos y \cdot 1) - \cos y]$$

$$+ 0 . [-x\sin y - y\cos y - \sin y].$$

$$= e^x [-x\cos y + y\sin y - \cos y - \cos y]$$

$$= e^x [-x\cos y + y\sin y - 2\cos y] \longrightarrow (2).$$

Adding (1) and (2), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x [2\cos y + x\cos y - y\sin y] + e^x [-x\cos y + y\sin y - 2\cos y]$$

$$= e^x [\cancel{2\cos y} + \cancel{x\cos y} - \cancel{y\sin y} - \cancel{x\cos y} + \cancel{y\sin y} - \cancel{2\cos y}]$$

$$= e^x [0]$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \,_{\shortparallel}$$

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3). If $f = \frac{1}{\sqrt{x^2+y^2+z^2}}$ show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

Solution :-

$$f = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$= (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial f}{\partial x} = -\frac{1}{\cancel{2}} (x^2+y^2+z^2)^{-\frac{1}{2}-1} . \cancel{2}x$$

$$= -x \, (x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 f}{\partial x^2} = -[x \cdot -3/2 (x^2+y^2+z^2)^{-3/2-1} \cdot 2x + (x^2+y^2+z^2)^{-3/2} \cdot 1]$$

$$= 3x^2 (x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2} \rightarrow (1)$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = 3y^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \rightarrow (2)$$

$$\frac{\partial^2 f}{\partial z^2} = 3z^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \rightarrow (3)$$

Adding (1), (2) and (3).

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 3x^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$+ 3y^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$+ 3z^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-5/2} (x^2+y^2+z^2)$$

$$- 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-5/2+1} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

4). If $r^2 = x^2 + y^2$ then show that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right].$$

Solution:

$$r^2 = x^2 + y^2$$

$$\cancel{2}r \frac{\partial r}{\partial x} = \cancel{2}x \qquad \cancel{2}r \frac{\partial r}{\partial y} = \cancel{2}y$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = y/r$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r.1 - x.\frac{\partial r}{\partial x}}{r^2} \qquad \frac{\partial^2 r}{\partial y^2} = \frac{r.1 - y\frac{\partial r}{\partial y}}{r^2}$$

$$= \frac{r - x.\frac{x}{r}}{r^2} \qquad = \frac{r - y\frac{y}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^2} \qquad = \frac{r^2 - y^2}{r^2}$$

$$= \frac{r^2 - x^2}{r^3} \longrightarrow (1) \qquad = \frac{r^2 - y^2}{r^3} \longrightarrow (2)$$

Adding (1) and (2), we get.

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r^2 - x^2 + r^2 - y^2}{r^3} = \frac{2r^2 - x^2 - y^2}{r^3}$$

$$= \frac{2r^2 - (x^2 + y^2)}{r^3}$$

$$= \frac{2r^2 - r^2}{r^3} = \frac{r^2}{r^3} = 1/r \longrightarrow (3)$$

$$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = 1/r\left[(x/r)^2 + (y/r)^2\right]$$

$$= 1/r\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right]$$

$$= 1/r\left[\frac{x^2 + y^2}{r^2}\right]$$

$$= 1/r\left[\frac{r^2}{r^2}\right]$$

$$= 1/r \longrightarrow (4)$$

From (3) and (4)

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = 1/r\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

(Q) If $u = \log r$ and $r^2 = (x-a)^2 + (y-b)^2$ Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution:-

$u = \log r \quad \therefore \frac{\partial u}{\partial r} = 1/r$

$r^2 = (x-a)^2 + (y-b)^2$

Diff. w.r. to x on both sides.

$$\cancel{2} r \frac{\partial r}{\partial x} = \cancel{2}(x-a)$$

$$\frac{\partial r}{\partial x} = \frac{x-a}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = 1/r \, \frac{x-a}{r} = \frac{x-a}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r^2(1) - (x-a)\, 2r \frac{\partial r}{\partial x}}{r^4}$$

$$= \frac{r^2 - (x-a)\, 2\cancel{r} \frac{x-a}{\cancel{r}}}{r^4}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r^2 - 2(x-a)^2}{r^4} \longrightarrow (1).$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{r^2 - 2(y-b)^2}{r^4} \longrightarrow (2).$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{r^2 - 2(x-a)^2}{r^4} + \frac{r^2 - 2(y-b)^2}{r^4}$$

$$= \frac{r^2 - 2(x-a)^2 + r^2 - 2(y-b)^2}{r^4}$$

$$= \frac{2r^2 - 2(x-a)^2 - 2(y-b)^2}{r^4}$$

$$= 2r^2 - 2[(x-a)^2 - (y-b)^2]/r^4$$

$$= \frac{2r^2 - 2r^2}{r^4} = 0/r^4 \therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(6). If $f = a\tan^{-1}(x/y)$. Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Solution :-

$$f = a\tan^{-1}(x/y) \qquad \tan^{-1}x = \frac{1}{1+x^2} - 1$$

$$\frac{\partial f}{\partial y} = a \cdot \frac{1}{1+\left(\frac{x^2}{y^2}\right)}\left(-\frac{x}{y^2}\right) = -\frac{a}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\left[\frac{(x^2+y^2)a - ax(2x)}{(x^2+y^2)^2}\right] = \frac{a(x^2-y^2)}{(x^2+y^2)^2} \rightarrow (1)$$

$$\frac{\partial f}{\partial x} = a \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot (1/y) = \frac{ay}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(x^2+y^2) \cdot a - ay(2y)}{(x^2+y^2)^2} = \frac{a(x^2-y^2)}{(x^2+y^2)^2} \rightarrow (2).$$

From (1), (2), we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

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(7). If $u = x^2y + y^2x + z^2x$ show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x+y+z)^2$$

Solution :-

$$u = x^2y + y^2z + z^2x$$

$$\frac{\partial u}{\partial x} = 2xy + z^2 \rightarrow (1).$$

$$\frac{\partial u}{\partial y} = x^2 + 2yz \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = y^2 + 2zx \rightarrow (3).$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=(2xy+z^2)+x^2+2yz+y^2+2zx$$

$$=x^2+y^2+z^2+2xy+2yz+2zx$$

$$=(x+y+z)^2$$

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(8). If $u=\tan^{-1} y/x$, using Euler's theorem, show that $x\,\partial u/\partial x+y\frac{\partial u}{\partial y}=0$.

Solution :-

$$u=\tan^{-1} y/x$$

$$\tan u=y/x=H_0$$

$$\sec^2 u\frac{\partial u}{\partial x}=\frac{\partial H_0}{\partial x}$$

$$x\sec^2 u\frac{\partial u}{\partial x}=x\frac{\partial H_0}{\partial x}\longrightarrow (1).$$

$$\sec^2 u\frac{\partial u}{\partial y}=\frac{\partial H_0}{\partial y}$$

$$y\sec^2 u\frac{\partial u}{\partial y}=y\frac{\partial H_0}{\partial y}\longrightarrow (2).$$

Adding (1) and (2), we get.

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=x\frac{\partial H_0}{\partial x}+y\frac{\partial H_0}{\partial y}$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=x\frac{\partial H_0}{\partial x}+y\frac{\partial H_0}{\partial y}.$$

$$=0.H_0 \text{ (by Euler's theorem)}$$

$$=0.$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

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(9). Verify Euler's theorem for the following function (i) y/x (ii) $\frac{1}{\sqrt{x^2+y^2}}$ (iii) $u = x^3+y^3+3x^2y-3xy^2$

Solution:

(i) Let $f = y/x$

This is a homogeneous function of degree 0.

Euler's theorem is $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

We have to verify that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = -y/x^2$ $\qquad$ $\frac{\partial f}{\partial y} = 1/x$

$\therefore x\frac{\partial f}{\partial x} = -y/x$ — (1) $\therefore y\frac{\partial f}{\partial y} = y/x$ — (2).

Adding (1) & (2), we get.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -y/x + y/x = 0$$

$$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0$$

$\therefore$ Euler's theorem is verified.

(ii) $f = \frac{1}{\sqrt{x^2+y^2}} = (x^2+y^2)^{-1/2}$

This is a homogeneous function of degree -1.

Euler's theorem is $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

We have to verify that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -f$.

$$\frac{\partial f}{\partial x} = -\cancel{1/2}\,(x^2+y^2)^{-1/2-1}\,\cancel{2}x$$

$$= -x(x^2+y^2)^{-3/2}$$

$$\therefore x\frac{\partial f}{\partial x} = -x^2(x^2+y^2)^{-3/2} \longrightarrow (1).$$

y^2

$$\frac{\partial f}{\partial y} = -\frac{1}{2}(x^2+y^2)^{-1/2-1}\, 2y$$

$$= -y(x^2+y^2)^{-3/2}$$

$$\therefore y\frac{\partial f}{\partial y} = -y^2(x^2+y^2)^{-3/2} \longrightarrow (2).$$

Adding (1) and (2), we get.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -x^2(x^2+y^2)^{-3/2} - y^2(x^2+y^2)^{-3/2}$$

$$= (x^2+y^2)^{-3/2}(-x^2-y^2).$$

$$= -(x^2+y^2)^{-3/2}(x^2+y^2)$$

$$= -(x^2+y^2)^{-1/2}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -f$$

$\therefore$ Euler's theorem is verified.

(iii). $u = x^3+y^3+3x^2y+3xy^2$.

$$\frac{\partial u}{\partial x} = 3x^2+6xy+3y^2$$

$$\frac{\partial u}{\partial y} = 3y^2+3x^2+6xy$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(3x^2+6xy+3y^2)+y(3y^2+3x^2+6xy)$$

$$= 3x^3+6x^2y+3xy^2+3y^3+3x^2y+6xy^2$$

$$= 3x^3+3y^3+9x^2y+9xy^2$$

$$= 3(x^3+y^3+3x^2y+3xy^2)$$

$$= 3u.$$

$\therefore$ Euler's theorem is verified.

Total differential coefficient

Theorem

Suppose $z = f(x, y)$ is a differential function of x and y where $x = g(t)$ and $y = h(t)$ are both differential function of t. Then z is a differential function of t and the total differential coefficient of z is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Proof:

We know that

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \rightarrow (1)$$

where $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$

Divide both sides of (1) by Δt

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

If $\Delta t \rightarrow 0$ then $\Delta x = g(t + \Delta t) - g(t) \rightarrow 0$ because g is differentiable and therefore continuous.

Similarly $\Delta y \rightarrow 0$;

$\therefore \varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$

$$\therefore \frac{dz}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\partial z}{\partial t}$$

$$= \frac{\partial z}{\partial x} \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t}$$

$$+ \varepsilon_1 \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \varepsilon_2 \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt}$$

$$= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Note :-

This is also called the chain rule for function of two variables.

Suppose $z = f(x, y)$ and each of x and y are function of two variables s and t.

i.e, $x = g(s, t)$ and $y = h(s, t)$

then z is a function of s and t.

Theorem :-

let $z = f(x, y)$ be a differentiable function of x and y where $x = g(x, t)$, $y = h(s, t)$ and the partial derivatives g_s, g_t, h_s and h_t exist.

Then $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

In general if u is a function of $x_1, x_2, \ldots x_n$ and each x_i is function of n variables $t_1, t_2, \ldots, t_n$ and if $\frac{\partial u}{\partial t_i}$ $(i = 1, 2, \ldots, m)$ exist.

Then $\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_2} + \ldots\ldots$

$+ \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$ $(i = 1, 2, \ldots m)$

— * — * — * —

Implicit functions :

Suppose an equation of the form $F(x,y) = 0$ defines y implicitly as a differentiable function of x, i.e, $y = f(x)$ where $F[x, f(x)] = 0$ for all x in the domain of f. If $F(x,y)$ is differentiable then by chain rule.

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0.$$

i.e., $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

i.e., $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Suppose $F(x,y,z) = 0$ and $z = f(x,y)$.
By chain rule we have.

$$\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

But $\frac{\partial}{\partial x}(x) = 1$ and $\partial/\partial x\,(y) = 0$.

$\therefore \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$

If $\frac{\partial F}{\partial z} \neq 0$, then $\frac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$.

Similarly. $\frac{\partial z}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$

—*—*—*—

Problems:

(1). If $u = e^x \sin y$ where $x = st^2$ and $y = s^2t$ find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

Solution:-

Given $u = e^x \sin y$, where $x = st^2$ and $y = s^2t$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \quad \& \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial x} = e^x \cdot 0 + e^x \sin y = e^x \sin y$$

$$\frac{\partial u}{\partial y} = e^x \cos y + \sin y \cdot 0 = e^x \cos y$$

$$\frac{\partial x}{\partial s} = s \cdot 0 + t^2 \cdot 1 = t^2, \quad \frac{\partial x}{\partial t} = s\, 2t + t^2 \cdot 0 = 2st$$

$$\frac{\partial y}{\partial s} = s^2 \cdot 0 + t \cdot 2s = 2st, \quad \frac{\partial y}{\partial t} = s^2 \cdot 1 + t^2 \cdot 0 = s^2$$

$$\frac{\partial u}{\partial s} = e^x \sin y \cdot t^2 + e^x \cos y\, 2st$$

$$= t^2 e^x \sin y + 2st\, e^x \cos y.$$

$$\frac{\partial u}{\partial t} = e^x \sin y \cdot 2st + e^x \cos y \cdot s^2$$

$$= 2st\, e^x \sin y + s^2 e^x \cos y.$$

—*—*—*—

(2). If $z = x^2+y^2$, $x = t^3$, $y = 1+t^2$ find $\frac{dz}{dt}$.

Solution :-

Given $z = x^2+y^2$, $x = t^3$, $y = 1+t^2$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

$\frac{\partial z}{\partial x} = 2x$ $\qquad$ $\frac{dx}{dt} = 3t^2$

$\frac{\partial z}{\partial y} = 2y$ $\qquad$ $\frac{dy}{dt} = 2t.$

$$\therefore \frac{dz}{dt} = 2x\cdot 3t^2 + 2y\cdot 2t$$
$$= 2(t^3)\,3t^2 + 2(1+t^2)\cdot 2t$$
$$= 6t^5 + 4t + 4t^3$$
$$= 6t^5 + 4t^3 + 4t.$$

(3). If $u = x^2+y^2+z^2$, $x = e^t$, $y = e^t\sin t$ and $z = e^t\cos t$ find $\frac{du}{dt}$.

Solution :-

Given $u = x^2+y^2+z^2$, $x = e^t$ $y = e^t\sin t$ & $z = e^t\cos t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{dx}{dt} = e^t$ $\qquad$ $\frac{dy}{dt} = e^t\cos t + e^t\sin t$

$= e^t(\cos t + \sin t)$

$\frac{\partial u}{\partial y} = 2y$

$\frac{\partial u}{\partial z} = 2z$ $\qquad$ $\frac{dz}{dt} = e^t(-\sin t) + e^t\cos t$

$= e^t\cos t - e^t\sin t$

$= e^t(\cos t - \sin t).$

$$\frac{du}{dt} = 2x \cdot e^t + 2y(e^t \sin t + e^t \cos t) + 2z(e^t \cos t - e^t \sin t).$$

$$= 2e^t [x + y(\sin t + \cos t) + z(\cos t - \sin t)]$$

$$= 2e^t [e^t + e^t \sin t (\sin t + \cos t) + e^t \cos t (\cos t - \sin t)]$$

$$= 2e^t [e^t + e^t \sin^2 t + e^t \cancel{\sin t \cos t} + e^t \cos^2 t - e^t \cancel{\sin t} \cos t]$$

$$= 2e^t [e^t + e^t (\sin^2 t + \cos^2 t)]$$

$$= 2e^t [e^t + e^t (1)]$$

$$= 2e^t [2e^t]$$

$$= 4e^{2t}$$

(1). If $u = \sin(x^2+y^2)$ where $a^2x^2 + b^2y^2 = c^2$ find $\frac{du}{dx}$.

Solution:

Given $u = \sin(x^2+y^2)$, where $a^2x^2+b^2y^2 = c^2$

$f(x,y) = a^2x^2 + b^2y^2 - c^2$.

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{\partial f}{\partial x} = a^2 2x$$

$$\frac{\partial f}{\partial y} = b^2 2y$$

$$= -\frac{\cancel{2}a^2x}{\cancel{2}b^2y}$$

$$\frac{\partial u}{\partial x} = \cos(x^2+y^2)(2x)$$

$$\frac{\partial u}{\partial y} = \cos(x^2+y^2)(2y).$$

$$= -\frac{a^2x}{b^2y}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dx}\right) + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= \frac{2b^2x\cos(x^2+y^2) - 2a^2x\cos(x^2+y^2)}{b^2}$$

$$= \frac{2x\left[b^2\cos(x^2+y^2) - a^2\cos(x^2+y^2)\right]}{b^2}$$

---

(2). If $z = f(y-z, z-x, x-y)$ show that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial z} = 0.$$

Solution :-

Given $z = f(y-z, z-x, x-y)$

Let $u = y-z$, $v = z-x$ $w = x-y$

$\therefore z = f(u, v, w)$

$\frac{\partial u}{\partial x} = 0$

$\frac{\partial v}{\partial x} = -1$

$\frac{\partial w}{\partial x} = 1$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1)$$

$$= -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \longrightarrow (1)$$

$\frac{\partial u}{\partial y} = 1$

$\frac{\partial v}{\partial y} = 0$

$\frac{\partial w}{\partial y} = -1$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1)$$

$$= \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w} \longrightarrow (2)$$

$$\frac{\partial z}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

$$\left\{ \frac{\partial u}{\partial z} = -1, \quad \frac{\partial v}{\partial z} = 1, \quad \frac{\partial w}{\partial z} = 0 \right.$$

$$= \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0)$$

$$= -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \longrightarrow (3).$$

Adding (1), (2), (3)

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial z} = -\cancel{\frac{\partial f}{\partial v}} + \cancel{\frac{\partial f}{\partial w}} + \cancel{\frac{\partial f}{\partial u}} - \cancel{\frac{\partial f}{\partial w}} - \cancel{\frac{\partial f}{\partial u}} + \cancel{\frac{\partial f}{\partial v}}$$

$$= 0$$

---

(2). If $u = f(x,y)$ and $x = X\cos\alpha - Y\sin\alpha$

$y = X\sin\alpha + Y\cos\alpha$ Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$= \frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}$.

Solution :-

Given $u = f(x,y)$ and $x = X\cos\alpha - Y\sin\alpha$

$y = X\sin\alpha + Y\cos\alpha$.

$$\frac{\partial u}{\partial X} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial X} \quad \& \quad \frac{\partial u}{\partial Y} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial Y}.$$

$$\frac{\partial x}{\partial X} = \cos\alpha \qquad \frac{\partial x}{\partial Y} = -\sin\alpha$$

$$\frac{\partial y}{\partial X} = \sin\alpha \qquad \frac{\partial y}{\partial Y} = \cos\alpha$$

$$\therefore \frac{\partial u}{\partial X} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha \longrightarrow (1)$$

and

$$\therefore \frac{\partial u}{\partial Y} = \frac{\partial u}{\partial x}(-\sin\alpha) + \frac{\partial u}{\partial y}\cos\alpha \longrightarrow (2)$$

i.e, $\frac{\partial u}{\partial x} = \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y}\right)u$

and

$$\frac{\partial u}{\partial y} = \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right)u$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y}\right)\left(\cos\alpha \frac{\partial u}{\partial x} + \sin\alpha \frac{\partial u}{\partial y}\right).$$

$$= \cos^2\alpha \frac{\partial^2 u}{\partial x^2} + \cos\alpha \sin\alpha \frac{\partial^2 u}{\partial x \partial y} + \sin\alpha \cos\alpha \frac{\partial^2 u}{\partial y \partial x} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2} \longrightarrow (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(-\sin\alpha \, \partial/\partial x + \cos\alpha \, \partial/\partial y\right)\left(-\sin\alpha \frac{\partial u}{\partial x} + \cos\alpha \frac{\partial u}{\partial y}\right)$$

$$= \sin^2\alpha \frac{\partial^2 u}{\partial x^2} - \sin\alpha \cos\alpha \frac{\partial^2 u}{\partial x \partial y} - \cos\alpha \sin\alpha \frac{\partial^2 u}{\partial y \partial x} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2} \longrightarrow (4)$$

Adding (3) and (4).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cos^2\alpha \frac{\partial^2 u}{\partial x^2} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2} + \sin^2\alpha \frac{\partial^2 u}{\partial x^2} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial x^2}\underbrace{(\sin^2\alpha + \cos^2\alpha)}_{(1)} + \frac{\partial^2 u}{\partial y^2}\underbrace{(\sin^2\alpha + \cos^2\alpha)}_{(1)}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \,//$$

H.W

① If $u = (x-y)(y-z)(z-x)$ show that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$. (ii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$

Solution:

Given $u = (x-y)(y-z)(z-x)$

$$\therefore \frac{\partial u}{\partial x} = (y-z)[(x-y)(-1) + (z-x)-1]$$

$$= (y-z)(-x+y+z-x)$$

$$= (y-z)(y+z) - 2(y-z)x$$

$$\frac{\partial u}{\partial x} = y^2 - z^2 - 2yx + 2zx$$

Similarly. $\frac{\partial u}{\partial y} = z^2 - x^2 - 2zy + 2xy$

$$\frac{\partial u}{\partial z} = x^2 - y^2 - 2xz + 2yz$$

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{y^2} - \cancel{z^2} - \cancel{2yx} + \cancel{2zx} + \cancel{z^2} - \cancel{x^2} - \cancel{2zy}$
$+ \cancel{2xy} + \cancel{x^2} - \cancel{y^2} - \cancel{2xz} + \cancel{2yz}$

$= 0$ ॥

(ii). u is a homogeneous function of degree 3.

Since.

$$u(xt, yt, zt) = (tx - ty)(ty - tz)(tz - tx)$$

$$= t^3(x-y)(y-z)(z-x)$$

$$= t^3 u(x, y, z)$$

So, by Euler's theorem, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u.$$

Example:

If $\sin u = \frac{x^2y^2}{x+y}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

Solution:

Given $\sin u = \frac{x^2y^2}{x+y}$

Let $f(x,y) = \frac{x^2y^2}{x+y}$

$$\therefore f(tx,ty) = \frac{t^2x^2t^2y^2}{tx+ty}$$

$$= \frac{t^4(x^2 y^2)}{t(x+y)}$$

$$= t^3 \cdot \frac{x^2+y^2}{x+y}$$

$$f(tx,ty) = t^3 f(x,y)$$

$\therefore f$ is a homogeneous function of degree 3 in x, y.

$\cancel{\Rightarrow \tan u = \frac{x^3+y^3}{x+y}}$

By Euler's theorem, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f \quad \therefore (f = \sin u).$$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 3\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 3\sin u$$

$\div \cos u$
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u \;//$$

Example :-

If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, ~~there~~ prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u.$$

Solution :-

Given $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\sin u = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \sin u.$$

$$\therefore f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}$$

$$= \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})} = t^{1/2}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$$

$$f(tx, ty) = t^{1/2} f(x,y)$$

$\therefore f$ is a homogeneous function of degree $1/2$ in x, y.

By Euler's theorem, we get.

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{1}{2} f$$

$$x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) = \frac{1}{2} \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial x} = \frac{1}{2} \sin u$$

$\div \cos u$
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} = \frac{1}{2} \frac{\sin u}{\cos u} \qquad \left(\because \frac{\sin u}{\cos u} = \tan u\right)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} = \frac{1}{2} \tan u.$$

Example :-

If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Solution :-

Given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$.

$$\tan u = \frac{x^3+y^3}{x-y}.$$

$$f(x,y) = \frac{x^3+y^3}{x-y} = \tan u$$

$$f(tx,ty) = \frac{t^3x^3+t^3y^3}{tx-ty} = \frac{t^3(x^3+y^3)}{t(x-y)}$$

$$= t^2 \cdot \frac{x^3+y^3}{x-y}$$

f is a homogeneous function of degree 2 in x, y.

By, Euler's theorem, we get.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f.$$

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cdot\frac{\sin u}{\cos u}\cdot\cos^2 u$$

$$= \sin 2u$$

Example :

If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0.$$

Solution :-

Given $u(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$$u(tx,ty,tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx}$$

$$= t^0\left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right] =$$

$$= t^0\, u(x,y,z).$$

$\therefore$ u is a homogeneous function of degree 0 in x, y, z.

By. Euler's theorem, we get.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

5m, 10m Exam

If $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \sin 2u.$$

Solution :-

Given $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\Rightarrow \tan u = \frac{x^2+y^2}{x+y}$$

Let $f(x,y) = \frac{x^2+y^2}{x+y} = \tan u$

$$\therefore f(tx,ty) = \frac{t^2x^2+t^2y^2}{tx+ty} = \frac{t^2(x^2+y^2)}{t(x+y)}$$

$$= t\left(\frac{x^2+y^2}{x+y}\right)$$

$\therefore$ f is a homogeneous function of degree 1

By, Euler's theorem, we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1f$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f \Rightarrow x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial x}(\tan u)=\tan u$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=\tan u$$

$$\Rightarrow \sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\tan u}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\sin u}{\cos u}\Big/\frac{1}{\cos^2 u}$$

$$=\frac{\sin u}{\cancel{\cos u}}\times\frac{\cos^2 u}{1}$$

$$=\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u$$

Hints:

$$\sin 2A = 2\sin A\cos A$$

$$\frac{1}{2}\sin 2A=\sin A\cos A$$

Example :-

If $u = x^y$. then Show that (i) $U_{xy} = U_{yx}$

ii) $U_{xxy} = U_{xyx}$

Solution :-

Given $u = x^y$.

$\therefore u_x = y x^{y-1} \longrightarrow (1)$

and $\therefore U_y = x^y \log_e x$.

Differentiating (1) agin w.r to x,

$U_{xx} = y(y-1) x^{y-2} \longrightarrow (2)$.

Differentiating again w.r.to y. we get.

$U_{xxy} = y(y-1) x^{y-2} \log_e x$
$+ x^{y-2} [y.1 + (y-1).1]$.

$U_{xxy} = x^{y-2} [y(y-1) \log_e x + 2y - 1] \rightarrow (3)$.

Differentiating (1) w.r.to y, we get.

$U_{xy} = y . x^{y-1} \log_e x + x^{y-1} 1$.

$\Rightarrow U_{xy} = x^{y-1} [1 + y \log_e x] \rightarrow (4)$

Differentiating (2) w.r.to x, we get.

$U_{yx} = x^y . 1/x + \log_e x . y x^{y-1}$

$\Rightarrow U_{yx} = x^{y-1} (1 + y \log_e x]. \rightarrow (5)$.

From (4) and (5), we get $\therefore U_{xy} = U_{yx}$

Again differentiating (4) w.r.to x. we get

$$U_{xyx} = x^{y-1}\left[y \cdot \frac{1}{x}\right] + (1+y\log_e x)\cdot(y-1)x^{y-2}$$

$$= x^{y-2}\cdot y + (y-1)(1+y\log_e x)x^{y-2}$$

$$U_{xyx} = x^{y-2}\left[y + (y-1)(1+y\log_e x)\right]$$

$$U_{xyx} = x^{y-2}\left[y(y-1)\log_e x + 2y-1\right] \rightarrow (6)$$

From (3) and (6) we get (ii) $\therefore U_{xxy} = U_{xyx}$.

H.W.

(2). If $u = \sin^{-1} \frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}$ prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}\tan u$$

Solution:

Given $u = \sin^{-1} \frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}$

$$\sin u = \frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}$$

Let $f(x,y) = \frac{x^3+y^3}{\sqrt{x}+\sqrt{y}} = \sin u$

$$f(tx,ty) = \frac{t^3x^3+t^3y^3}{\sqrt{tx}+\sqrt{ty}} \Rightarrow \frac{t^3(x^3+y^3)}{\sqrt{t}(\sqrt{x}+\sqrt{y})}$$

$$= \frac{t^3(x^3+y^3)}{t^{1/2}(\sqrt{x}+\sqrt{y})}$$

$= 3-\frac{1}{2}$
$= \frac{6-1}{2}$
$= 5/2$

$$= t^{5/2}\,\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}$$

$\therefore f$ is a homogeneous function of degree $5/2$ in x, y.

By. Euler's theorem. we get.

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 5/2 \, f$$

$$x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) = 5/2 \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial x} = 5/2 \sin u$$

$\div \cos u$ $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} = 5/2 \frac{\sin u}{\cos u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} = 5/2 \tan u$$

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8. If $u = \tan^{-1} y/x$ using Euler's theorem.

H.W Show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

Solution :-

Given $u = \tan^{-1} y/x$

$\tan u = y/x$

Let $f(x,y) = y/x = \tan u$

$$f(tx, ty) = \frac{ty}{tx} \Rightarrow t^0 (y/x)$$

$\therefore$ f is a homogeneous function of degree 0.

By. Euler's theorem, we get,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0 f$$

$$x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = 0 (\tan u)$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 0$$

$$\sec^2 u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = 0 \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{0}{\sec^2 u}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

4. Verify Euler's theorem in the following:

(i) $u = \sin \frac{x^2+y^2}{xy}$

Given :-

$$u = \sin \frac{x^2+y^2}{xy}$$

$$H_0 = 1$$

$$\sin u = H_1 \longrightarrow (1)$$

eq (1) Differentiating w.r to x

$$\cos u \cdot \frac{\partial u}{\partial x} = \frac{\partial H_1}{\partial x}$$

$$x \cos u \cdot \frac{\partial u}{\partial x} = x \cdot \frac{\partial H_1}{\partial x} \longrightarrow (2).$$

eq (1) D.w.r. to y

$$\cos u \frac{\partial u}{\partial y} = \frac{\partial H_1}{\partial y}$$

$$y \cos u \frac{\partial u}{\partial y} = y \frac{\partial H_1}{\partial y} \longrightarrow (3).$$

(2) + (3) $\Rightarrow$

$$x \cos u \cdot \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = x \frac{\partial H_1}{\partial x} + y \frac{\partial H_1}{\partial y}$$

$$\cos u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right] = 1 . H_1$$

$$\cos u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right] = 1 . \sin u .$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$$

iii). $u = \frac{x-y}{x+y}$.

$\frac{vu' - uv'}{v^2}$

Given :-

$u = \frac{x-y}{x+y}$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$

$u = \frac{x-y}{x+y} = Ho$

$\frac{\partial u}{\partial x} = \frac{(x+y)(1-0) - (x-y)(1-0)}{(x+y)^2}$

$= \frac{(x+y) - (x-y)}{(x+y)^2}$

$= \frac{\cancel{x}+y-\cancel{x}+y}{(x+y)^2}$

$\frac{\partial u}{\partial x} = \frac{2y}{(x+y)^2}$

$\frac{\partial u}{\partial y} = \frac{(x+y)(0-1) - (x-y)(0+1)}{(x+y)^2}$

$= \frac{(x+y) - (x-y)}{(x+y)^2}$

$= \frac{\cancel{x}+y-\cancel{x}+y}{(x+y)^2}$

$\frac{\partial u}{\partial y} = 2y/(x+y)^2$

$\therefore x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 0$

$\frac{2xy}{(x+y)^2} - \frac{2xy}{(x+y)^2} = 0$

$\therefore 0 = 0$,

(iii). $U = x^3 \cos(y/x)$

Solution:

Given $U = x^3 \cos(y/x)$

diff partially w.r.t 'x'

$$\frac{\partial U}{\partial x} = 3x^{2} \cos(y/x) + x^{3}\left(-\sin(y/x) - \frac{y}{x^2}\right)$$

$$= 3x^{2} \cos(y/x) + xy \sin(y/x)$$

$$x \frac{\partial U}{\partial x} = 3x^{3} \cos(y/x) + x^2 y \sin(y/x) \rightarrow (1)$$

Diff. partially w.r. to 'y'.

$$\frac{\partial U}{\partial y} = x^{2}\left[-\sin(y/x) \cdot \frac{1}{x}\right]$$

$$= -x^2 \sin(y/x)$$

$$y \cdot \frac{\partial U}{\partial y} = -x^2 y \sin(y/x) \rightarrow (2)$$

(1) + (2)

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 3x^3 \cos(y/x) + x^2 y \sin(y/x) - x^2 y \sin(y/x)$$

$$= 3x^3 \cos(y/x)$$ Given Sum

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 3U$$ //