# 19UMA05

# STATICS

## B.Sc. MATHEMATICS

## $\hat{\text{III}}$ - SEMESTER

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# CORE - V - STATICS

**Unit - I :**

Parallelogram law of forces - Triangular law of forces - Perpendicular triangular forces - Converse of the triangular law of forces - The polygon of forces - Lami's theorem - Like and unlike parallel forces - Problems - Moments - Definition - Varignon's theorem - Problems (Chapter II (sections 1 to 9), chapter III (sections 1 to 12).)

**Unit - II :**

Couples - Moments of a couple - Theorems on couples - Problems. (chapter IV (section 1 to 10).).

**Unit - III :**

Friction : Introduction - Experimental results - Statical and Dynamical limiting friction - coefficients of friction - angle of friction - Equilibrium of a particle on rough inclined plane - Equilibrium of a particle on a rough inclined plane under a force parallel to the plane - Equilibrium of a particle on a rough inclined plane under any force parallel to the plane - Equilibrium of a

Particle on a rough inclined plane under any force. - Problems. (Chapter VII (section 1 - to 12)).

**Unit - IV :**

Centre of gravity : Centre of like parallel forces - centre of gravity - Distinction between centre of gravity and centre of mass - centre of gravity of a body is unique - Determination of centre of gravity in simple cases - centre of gravity by symmetry - C.G. of a uniform triangular lamia - Theorem - C.G. of 3 rods forming a triangle - General formula for determination of C.G. of a trapezium - Problems. (Chapter VIII (sections 1 to 13)).

**Unit - V**

Virtual Work :- Work - Theorem - Method of Virtual work - Principle of virtual work for a system of coplanar forces acting on a body - Forces which may be omitted in forming the equation of virtual work - Work done by an extensible string - Work done by the weight of the body - Application of the principle of virtual work - Problems.

# UNIT - I

Parallelogram law of forces - Triangular law of forces - Perpendicular triangular law of forces - The polygon of forces - Converse of the triangular law of forces - Like and unlike parallel forces - Problems - Moments - Definition - Varignon's theorem - Problem [Chapter II (section 1 to 9) chapter III (section 1 to 12)].

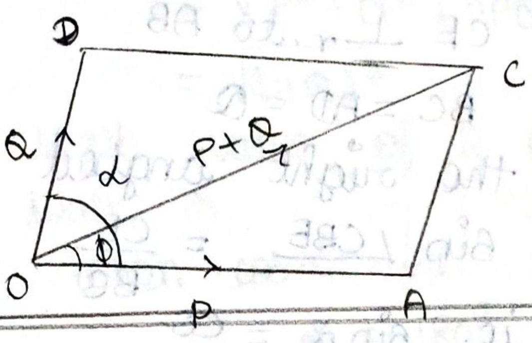
Unit - I :- STATICS

Force :-

Force is defined as any cause which produces or tends to produce a change in the existing state of rest of a body or of its uniform motion in a straight

Resultant of two forces :-

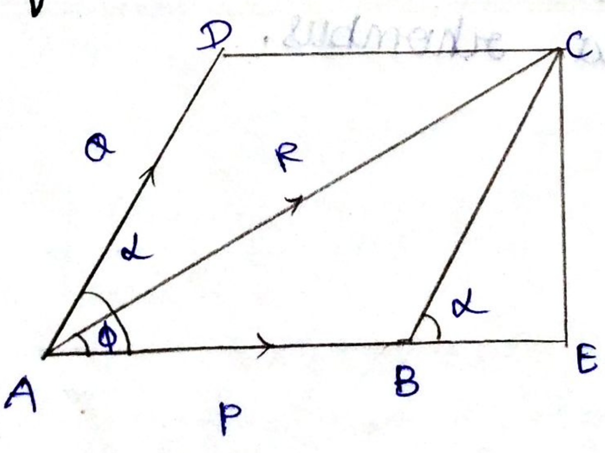
Let the forces P and Q act on a particle at O. Then their resultant is P+Q.



1). Parallelogram Law of Forces :-

If two forces acting at a point be represented in magnitude and direction by the sides of a parallelogram drawn from the point. Their resultant is represented both in magnitude and direction by the diagonal of the parallelogram drawn through that point.

Proof :-



Let the two forces P and Q acting at A be represented by AB and AD and let the angle between them be $\alpha$.

i.e., $\angle BAD = \alpha$

complete the parallelogram BADC then the diagonal AC will represent the resultant.

Let R be the Magnitude of the resultant and let it make an angle $\phi$ with P

i.e., $\angle CAB = \phi$

Draw $CE \perp^r$ to AB

$BC = AD = Q$

From the right angled $\triangle CBE$,

$\sin \angle CBE = \frac{CE}{BC}$

ie $\sin\alpha = \frac{CE}{BC}$

$\sin\alpha = \frac{CE}{Q}$

$CE = Q \sin\alpha \rightarrow ①$

$\cos\alpha = \frac{BE}{BC} = \frac{BE}{Q}$

$BE = Q\cos\alpha \rightarrow ②$

$R^2 = AC^2 = AE^2 + CE^2$

$= (AB + BE)^2 + CE^2$

In this case, the Parallelogram becomes a rhombus.

Corollary : 3

Let the magnitudes P and Q of two forces acting at an angle $\alpha$ be given.

Then their resultant R is greatest when $\cos\alpha$ is greatest.

ie., when $\cos\alpha = 1$

$$\alpha = 0^\circ$$

In This case, the forces act along the same line in the same direction.

and
$$R = \sqrt{P^2+Q^2+2PQ(1)}$$
$$= \sqrt{P^2+Q^2+2PQ}$$
$$= \sqrt{(P+Q)^2}$$

The least value of R occurs when $\cos\alpha$ is least

ie., when $\cos\alpha = -1$

$$\alpha = 180^\circ$$

In this case, the forces ~~ab~~ ~~da~~ act along the same line but it the opposite direction and

$$R = \sqrt{P^2+Q^2+2PQ(-1)}$$
$$= \sqrt{P^2+Q^2-2PQ}$$
$$= \sqrt{(P-Q)^2} \qquad = \sqrt{(P-Q)^2}$$
$$R = P-Q$$

Problems :-

1). The resultant of two forces P, Q acting at a certain angle is x and that of P, R acting at the same angle is also x. The resultant of Q, R again acting at the same angle is y. Prove that

$$P = (x^2 + QR)^{1/2} = \frac{QR(Q+R)}{Q^2+R^2-y^2}$$

| P | Q | R |
|---|---|---|
| P | Q | x |
| P | R | x |
| Q | R | y |

Prove that, if $P+Q+R=0$, $y=x$

Soln :-

Let P and Q act at an angle $\alpha$ from the data. We have the following results

$$x^2 = P^2 + Q^2 + 2PQ\cos\alpha \rightarrow (1)$$
$$x^2 = P^2 + R^2 + 2PR\cos\alpha \rightarrow (2)$$
$$y^2 = Q^2 + R^2 + 2QR\cos\alpha \rightarrow (3)$$

$(1)-(2) \Rightarrow 0 = Q^2 - R^2 + 2PQ\cos\alpha - 2PR\cos\alpha$

$$0 = Q^2 - R^2 + 2P\cos\alpha(Q-R)$$

$\div (Q-R)$ $\quad 0 = (Q+R)\cancel{(Q-R)} + 2P\cos\alpha\cancel{(Q-R)}$

$$0 = Q+R+2P\cos\alpha.$$
$$2P\cos\alpha = -Q-R$$
$$\cos\alpha = \frac{-(Q+R)}{2P} \rightarrow (4)$$

Using (4) in (1), we get

$$x^2 = P^2+Q^2+\cancel{2P}Q\left[\frac{-(Q+R)}{\cancel{2P}}\right]$$
$$x^2 = P^2+Q^2-Q(Q+R)$$
$$x^2 = P^2 \cancel{+Q^2} - \cancel{Q^2} - \cancel{R}\,QR$$

$x^2 = P^2 - QR$

$\Rightarrow P^2 = x^2 + QR$

$\Rightarrow P = (x^2 + QR)^{1/2}$

Using (4) in (3), we get

$$y^2 = Q^2 + R^2 + 2QR\left[\frac{-(Q+R)}{2P}\right]$$

$$y^2 = Q^2 + R^2 - \frac{QR(Q+R)}{P}$$

$$\Rightarrow \frac{QR(Q+R)}{P} = Q^2 + R^2 - y^2$$

$$\Rightarrow \frac{QR(Q+R)}{Q^2+R^2-y^2} = P$$

If $P+Q+R = 0$ then $Q+R = -P$

From eqn (4)

$$\cos\alpha = -\frac{(-P)}{2P} = \frac{P}{2P} = 1/2$$

$$\cos\alpha = 1/2 \rightarrow (5)$$

Using (5) in (2) and (3), we get.

$$x^2 = P^2 + R^2 + 2PR(1/2)$$

(ie.,) $x^2 = P^2 + R^2 + PR \rightarrow (6)$

$$y^2 = Q^2 + R^2 + 2QR(1/2)$$

(ie.,) $y^2 = Q^2 + R^2 + QR \rightarrow (7)$

(6) - (7) $x^2 - y^2 = P^2 + R^2 + PR - Q^2 - R^2 - QR$

$$= P^2 + PR - Q^2 - QR$$

$$= P^2 - Q^2 + PR - QR$$

Other
$\div P-Q$

$$= (P^2 - Q^2) + R(P-Q)$$

$$= (P+Q)(P-Q) + R(P-Q)$$

$$= (P+Q+R)(P-Q)$$

$$= 0(P-Q) = 0$$

$x^2 - y^2 = 0 \Rightarrow x^2 = y^2 \Rightarrow \boxed{x = y}$

2). If the resultant of forces 3P, 5P is equal to 7P find (i) The angle between the force (ii) The angle which the resultant makes with the first forces.

Soln:-

We know that the resultant of two forces ~~pand~~ P and Q is R then

$$R^2 = P^2+Q^2+2PQ\cos\alpha$$

Here $P \Rightarrow 3P$

$Q \Rightarrow 5P$

$R \Rightarrow 7P$

$$(7P)^2 = (3P)^2+(5P)^2+2(3P)(5P)\cos\alpha$$

$$49P^2 = 9P^2+25P^2+30P^2\cos\alpha$$

$$49P^2-34P^2 = 30P^2\cos\alpha$$

$$15P^2 = 30P^2\cos\alpha$$

$$\cos\alpha = \frac{15P^2}{30P^2}$$

$$\cos\alpha = 1/2$$

$$\alpha = \cos^{-1}(1/2)$$

$$\alpha = 60^\circ$$

If $\phi$ is the angle which the resultant R makes with the first forces P then

$$\tan\phi = \frac{Q\sin\alpha}{P+Q\cos\alpha}$$

$$= \frac{5P\sin\alpha}{3P+5P\cos\alpha} = \frac{5P\sin 60^\circ}{3P+5P\cos 60^\circ}$$

$$= \frac{5P(\sqrt{3}/2)}{3P+5P(1/2)}$$

$$= \frac{5P\sqrt{3}/2}{3P + 5P/2}$$

$$= \frac{5P\sqrt{3}/2}{\frac{6P+5P}{2}}$$

$$= \frac{5P\sqrt{3}/2}{11P/2} = \frac{5P\sqrt{3}}{11P} = \frac{5\sqrt{3}}{11}$$

$$\phi = \tan^{-1}\left(\frac{5\sqrt{3}}{11}\right)$$

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3). The resultant of two forces 3P and 2.P. is R. If the first force is doubled, the resultant is also doubled. Find the angle between the forces.

Soln :-

let $\alpha$ be the angle between the forces.

| P | Q | R |
|---|---|---|
| 3P | 2P | R |

We know that

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

$$R^2 = (3P)^2 + (2P)^2 + 2(3P)(2P)\cos\alpha$$

$$R^2 = 9P^2 + 4P^2 + 12P^2\cos\alpha$$

$$R^2 = 13P^2 + 12P^2\cos\alpha \longrightarrow (1)$$

If the first force is doubled. Then the resultant is also doubled.

We have $(2R)^2 = (6P)^2 + (2P)^2 + 2(6P)(2P)\cos\alpha$

$$4R^2 = 36P^2 + 4P^2 + 24P^2\cos\alpha$$

$$4R^2 = 40P^2 + 24P^2\cos\alpha$$

$\div 4$ $\quad R^2 = 10P^2 + 6P^2\cos\alpha \rightarrow$ ②

from eqn ① and ②, we get

$$13P^2 + 12P^2\cos\alpha = 10P^2 + 6P^2\cos\alpha$$

$$13P^2 - 10P^2 = 6P^2\cos\alpha - 12P^2\cos\alpha$$

$$3P^2 = -6P^2\cos\alpha$$

$$\cos\alpha = \frac{3P^2}{-6P^2}$$

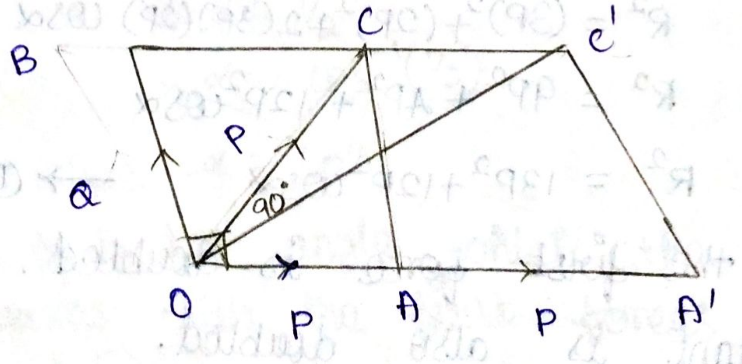
$$\cos\alpha = -1/2$$

$$\alpha = \cos^{-1}(-1/2)$$

$$\alpha = 120^\circ$$

4). The resultant of two forces P and Q is of magnitude P. Show that if the force P is doubled Q remaining unaltered. Then the new resultant will be at right angles to Q and its magnitude will be $\sqrt{4P^2 - Q^2}$.

Soln:-




Let OA and OB represent P and Q

Then OC is their resultant so that OC = OA = P.

Let us Produce OA to A' such that

OA = AA' = P and OC' = R.

which is the resultant of 2P (= OA') and Q (= OB)

$OC' = AA' = OC$

Hence $\angle BOC' = 90°$

Let us Produce OA to A' such that

∴ From the right angle $\triangle BOC'$

$$OC'^2 = BC'^2 - OB^2$$

$$R^2 = (2P)^2 - Q^2$$

$$R^2 = 4P^2 - Q^2$$

$$\Rightarrow R = \sqrt{4P^2 - Q^2}$$

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5). The resultant of force P and Q is R. If Q is doubled then R is doubled. R is also doubled if Q is reversed. Show that

$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$

Soln:-

We know that the resultant of two forces P and Q is R then

$$R^2 = P^2 + Q^2 + 2PQ \cos\alpha$$

∴ From the given data

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \rightarrow ①$$

$$4R^2 = P^2 + 4Q^2 + 4PQ\cos\alpha \rightarrow ②$$

$$4R^2 = P^2 + Q^2 - 2PQ\cos\alpha \rightarrow ③$$

$$① + ③ \Rightarrow 5R^2 = 2P^2 + 2Q^2$$

$$2P^2 + 2Q^2 - 5R^2 = 0 \rightarrow ④$$

Multiply eqn ③ by ② and then adding it to eqn ②, we get

③ × ②

③ + ②

$$12R^2 = 3P^2 + 6Q^2$$

$\div 3$

$$P^2 + 2Q^2 = 4R^2$$

$$P^2 + 2Q^2 - 4R^2 = 0 \rightarrow ⑤$$

Solving eqn ④ and ⑤, we get

| $P^2$ | $Q^2$ | $R^2$ | $P^2$ | $Q^2$ |
|---|---|---|---|---|
| 2 | 2 | -5 | 2 | 2 |
| 1 | 2 | -4 | 1 | 2 |

(or)

| $P^2$ | $Q^2$ | $R^2$ |
|---|---|---|
| 2 | 2 | -5 |
| 1 | 2 | -4 |

$$\frac{P^2}{-8+10} = \frac{Q^2}{-5+8} = \frac{R^2}{4-2}$$

$$\frac{P^2}{2} = \frac{Q^2}{3} = \frac{R^2}{2}$$

$$P^2 : Q^2 : R^2 = 2 : 3 : 2$$

$$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$$

H.

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6. The resultant of two forces P and Q acting at an angle $\theta$ is equal to $(2m+1)\sqrt{P^2+Q^2}$; when they act at an angle $90° - \theta$, the resultant is $(2m-1)\sqrt{P^2+Q^2}$

Prove that $\tan\theta = \frac{m-1}{m+1}$

Solns:-

We know that the resultant of two forms P and Q acting at an angle $\alpha$ is

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

$\therefore$ From the given data, we have the resultant of P and Q acting at angle $\theta$ is

$$(2m+1)^2 (P^2+Q^2) = P^2+Q^2 + 2PQ\cos\theta$$

$$(2m+1)^2 (P^2+Q^2) - P^2+Q^2 = 2PQ\cos\theta$$

$$[(2m+1)^2 - 1]\, P^2+Q^2 = 2PQ\cos\theta$$

$$[(4m^2+4m+1) - 1]\, P^2+Q^2 = 2PQ\cos\theta$$

$$(4m^2+4m)(P^2+Q^2) = 2PQ\cos\theta \rightarrow ①$$

The resultant of P and Q acting at angle $90°-\theta$ is

$$(2m-1)^2 (P^2+Q^2) = P^2+Q^2 + 2PQ\cos(90°-\theta)$$

$$(2m-1)^2 (P^2+Q^2) - (P^2+Q^2) = 2PQ\sin\theta$$

$$(4m^2-4m+1-1)(P^2+Q^2) = 2PQ\sin\theta$$

$$(4m^2-4m)(P^2+Q^2) = 2PQ\sin\theta \rightarrow ②$$

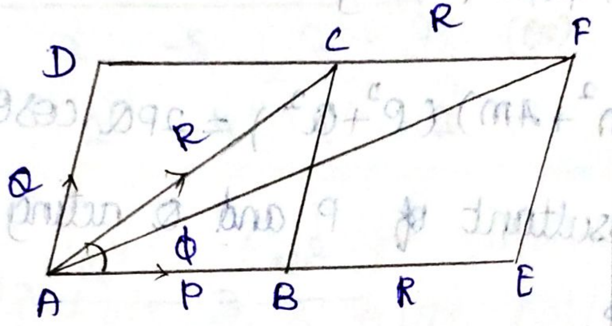
$$\frac{②}{①} \Rightarrow \frac{\cancel{2PQ}\sin\theta}{\cancel{2PQ}\cos\theta} = \frac{(4m^2-4m)\cancel{(P^2+Q^2)}}{(4m^2+4m)\cancel{(P^2+Q^2)}}$$

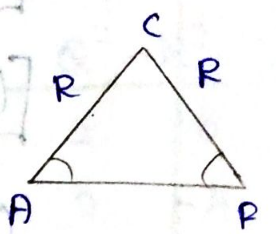
$$\tan\theta = \frac{4m^2-4m}{4m^2+4m} = \frac{\cancel{4m}(m-1)}{\cancel{4m}(m+1)}$$

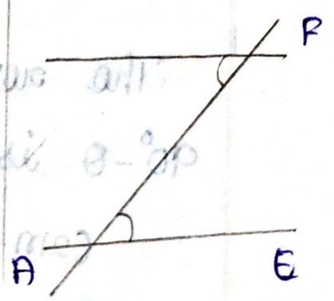
$$\tan\theta = \frac{m-1}{m+1}$$

7. If the resultant R of two forces P and Q inclined to one another at any given angle makes an angle $\phi$ with the direction of P, show that the resultant of forces (P+R) and Q acting at the same angle will make an angle $\phi/2$ with the direction of P+R.

solns:-




Let $\overline{AB} = P$ and $\overline{AD} = Q$

From the Parallelogram ABCD,

$\overline{AB} + \overline{AD} = \overline{AC} = R$

To mark the force P+R;

Produce AB to E so that

BE = AC

In the Parallelogram DAEF, $\overline{AF}$ given the new resultant,

In $\triangle$ CAF, CA = CF (each representing R in magnitude)

$\therefore \angle CAF = \angle CFA$

$= \angle FAE$ (Alternate angles).

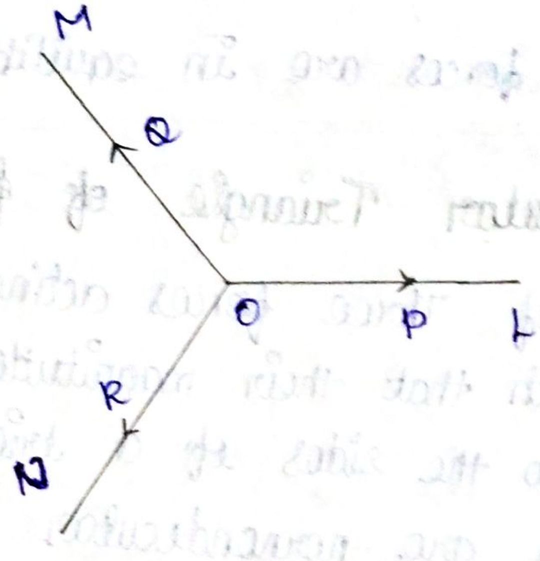
ie., $\angle CAF = \angle FAE$

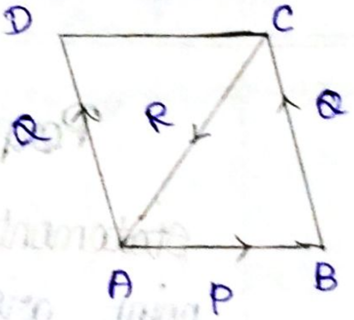
$\Rightarrow$ AF bisides $\angle CAB$
(bisects)

$\therefore \angle CAF = \angle FAE = \frac{\phi}{2}$

## Triangle of Forces

If three forces acting at a point canbe represented in magnitude and direction by the sides of a triangle takes in order, they willbe in equilibrium.




Let the forces P, Q, R act at a point O and be represented in magnitude and direction by the sides AB, BC, CA of the triangle ABC. We have to prove that they will be in equilibrium.

complete the Parallelogram BADC as AD is equal and parallel to BC, AD also represents Q in magnitude and direction.

$P+Q = \overline{AB} + \overline{AD} = \overline{AC}$ (by Parallelogram Law)

This shows that the resultant of the forces P and Q at O is represented in magnitude and direction by AC

The third force R acts at O and it is represented in magnitude and direction by CA. Hence $P+Q+R = \overline{AC}$ at O + $\overline{CA}$ at O

$= \overline{AC} - \overline{AC}$

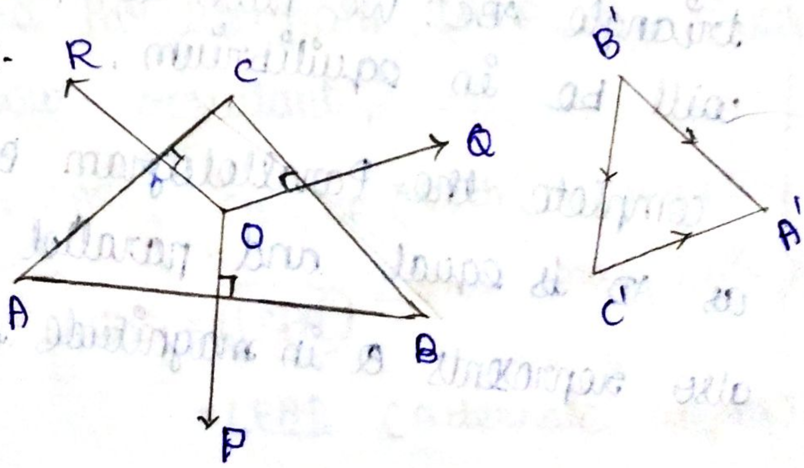
$= 0$ [as the two vector at O are equal and opposite.].

∴ The forces are in equilibrium.

## Perpendicular Triangle of forces

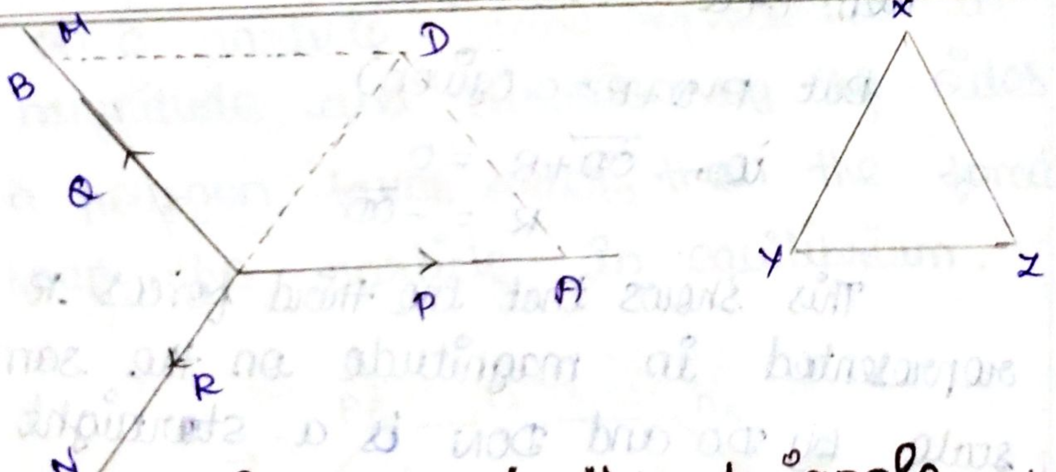
Statement :- If Three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards or all outwards, then also the forces will be in equilibrium.

Proof :-



Let the forces. P, Q, R meet at O. ABC is a triangle such that magnitudes of P, Q, R are proportional to the sides BC, CA and AB respectively of $\triangle ABC$ and their directions are perpendicular to the corresponding sides outwards. We have to prove that they will be in equilibrium. If we rotate the $\triangle ABC$ through $90°$ in its own plane, we will get a new triangle A', B', C' whose sides are parallel to the given forces and represent the forces both in magnitude and direction.

Hence by the triangle of forces P, Q, R are in equilibrium.




**Converse of the triangle of forces :-**

Statement :- If three forces acting at a point are in equilibrium, then any triangle drawn so as to have its sides parallel to the directions of the forces shall represent them in magnitude also

Let the three forces P, Q, R acting at O along the direction OL, OM and ON keep it in equilibrium. XYZ is a triangle such that the sides YZ, ZX and XY are parallel to the direction of P, Q, R respectively. We have to prove that the sides of $\Delta XYZ$ are proportional to the magnitudes of P, Q and R given that $P+Q+R=0$

Along OL, cut off OA to repres the magnitude of P on some scale

i.e., $\overline{OA} = P$

On the same scale, make $\overline{OB} = Q$

To get the resultant of P and Q. complete the parallelogram AOBD

Then $P+Q = \overline{OA} + \overline{OB} = \overline{OD}$

But $P+Q+R=0$ (given)

i.e., $\overline{OD} + R = 0$

$R = -\overline{OD}$

This shows that the third forces R is represented in magnitude on the same scale by DO and DON is a straight line.

Hence the three forces P, Q and R are parallel and proportional to the sides of the triangle OAD

Now any triangle like XYZ whose sides are parallel to the direction of

P, Q and R will be similar to $\Delta OAD$ and hence

$$\frac{YZ}{OA} = \frac{ZX}{AD} = \frac{XY}{DO}$$

But

$$\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{DO}$$

$$\frac{XZ}{P} = \frac{ZX}{Q} = \frac{XY}{R}$$

ie., The sides of $\Delta XYZ$ will be proportional to P, Q, R.

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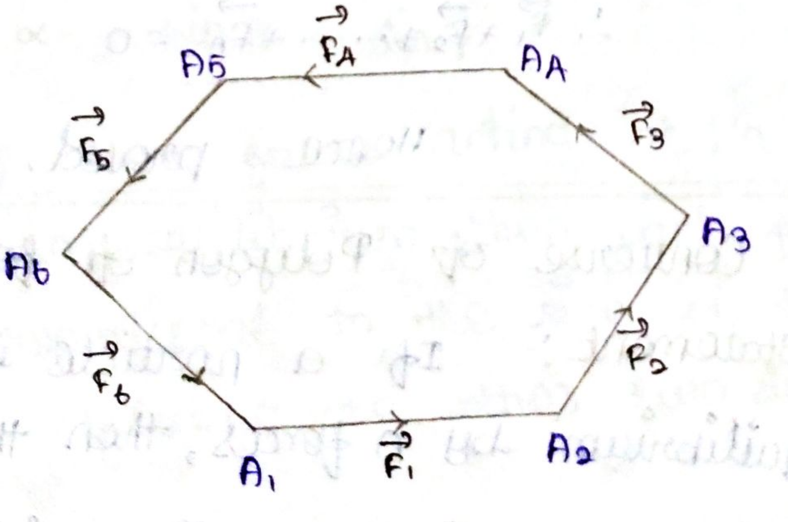
## Polygon of forces பலகோணம்

திசைப் பக்கங்களின் கோணங்களின் ஏமம்.

Statement :-

If several coplane forces, acting on a particle canbe represented in magnitude and direction by the sides of a polygon taken order., than the forces keep the particle in equilibrium.



Let the given forces be $\vec{F_1}, \vec{F_2}, \vec{F_3}, \ldots \vec{F_n}$ acting O can be represented in magnitude and direction by the sides

$A_1A_2, A_2A_3, A_3A_4, \ldots\ldots A_{n-1}A_n, A_nA$ of the Polygon $A_1A_2A_3\ldots\ldots A$.

To Prove that the forces keep the Particle in equilibrium.

Compounding the forces by vector

$$\vec{F_1} + \vec{F_2} = \overline{A_1A_2} + \overline{A_2A_3}$$

$$= A_1A_3$$

$$\vec{F_1} + \vec{F_2} + \vec{F_3} = \overline{A_1A_3} + \overline{A_3A_4}$$

$$= \overline{A_1A_4}$$

$$\vec{F_1} + \vec{F_2} + \ldots\ldots F_{n-1} = \overline{A_1A_{n-1}} + \overline{A_{n-1}A_n}$$

$$= \overline{A_1A_n} \rightarrow ①$$

$$\vec{F_1} + \vec{F_2} + \ldots\ldots + \vec{F_n} = A_1A_n \text{ at } O + A_nA_1 \text{ at } O$$

$$= \overline{A_1A_n} - \overline{A_nA_1} = 0$$

$$\therefore \vec{F_1} + \vec{F_2} + \ldots\ldots + \vec{F_n} = 0$$

Hence proved.

Converse of Polygon of forces:-

Statement :- If a particle is kept in equilibrium by n forces, then they can be represented by the sides of a n - sided polygon.

Same Diagram

Proof :-

In Particular consider 6 forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, \ldots \vec{F_6}$ which can be kept in equilibrium.

$$\vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} + \vec{F_5} + \vec{F_6} = 0$$

Take AB as $\vec{F_1}$, BC as $\vec{F_2}$, $\vec{CD}$ as $\vec{F_3}$, $\vec{DB}$ as $\vec{F_4}$, $\vec{EF}$ as $\vec{F_5}$, $\vec{FA}$ as $\vec{F_6}$

$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EF} + \vec{FA} = 0$$

Hence the proved.

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10marks

## Lami's Theorem

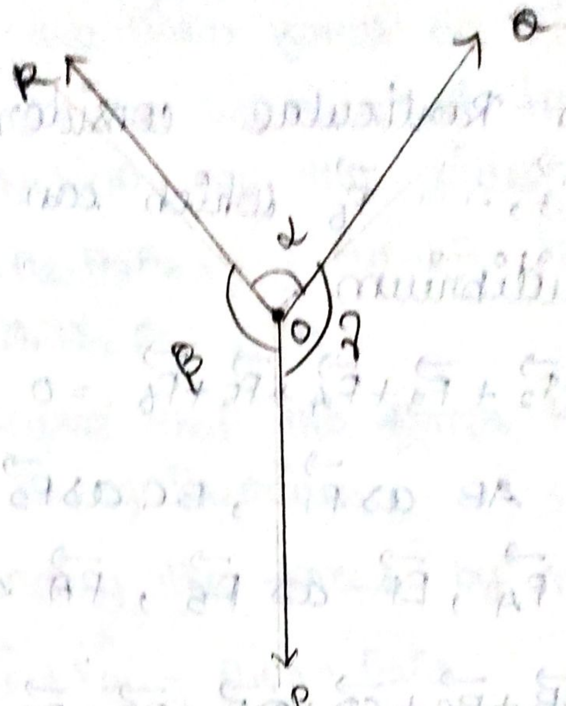
Statement :-

If a particle is in equilibrium under the action of forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ them to show that

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

If three forces acting at a point are in equilibrium then each force is proportional to the sine of the angle between the other two sides.

**Proof :-**




Let $\alpha$ be the angle between $\bar{R}$ and $\vec{Q}$

Let $\beta$ be the angle between $\vec{R}$ and $\vec{P}$

Let $\gamma$ be the angle between $\vec{P}$ and $\bar{Q}$

Since the three forces $\bar{P}, \bar{Q}$ and $\bar{R}$ acting at a point are in equilibrium. $\vec{P}+\bar{Q}+\bar{R}=0$ $\to (1)$

Multiply vectorially by $\bar{P}$, we get

$$(\bar{P}+\bar{Q}+\bar{R})\times\bar{P} = 0$$

$$\bar{P}\times\bar{P} + \bar{Q}\times\bar{P} + \bar{R}\times\bar{P} = 0$$

Let $\hat{n}$ be the unit vector $\perp_r$, to three forces such that $\bar{P}, \bar{Q}, \hat{n}$ form a right handed triad.

$$0 + QP\sin\gamma(-\hat{n}) + RP\sin\beta(\hat{n}) = 0$$

$$-QP\sin\gamma\,\hat{n} + RP\sin\beta\,\hat{n} = 0$$

$$R\cancel{P}\sin\beta\,\cancel{\hat{n}} = Q\cancel{P}\sin\gamma\,\cancel{\hat{n}}$$

$$R\sin\beta = Q\sin\gamma$$

$$\frac{R}{\sin\gamma} = \frac{Q}{\sin\beta} \to (2)$$

Again multiply eqn ① vectorially by $\bar{Q}$, we got

$$(\vec{P}+\bar{Q}+\bar{R}) \times \bar{Q} = 0$$

$$\bar{P}\times\bar{Q} + \bar{Q}\times\bar{Q} + \bar{R}\times\bar{Q} = 0$$

$$PQ \sin\gamma \, \hat{n} + 0 + RQ \sin\alpha (-\hat{n}) = 0$$

$$PQ \sin\gamma \, \hat{n} - RQ \sin\alpha \, \hat{n} = 0$$

$$P\cancel{Q} \sin\gamma \, \cancel{\hat{n}} = R\cancel{Q} \sin\alpha \, \cancel{\hat{n}}$$

$$P \sin\gamma = R \sin\alpha$$

$$\frac{P}{\sin\alpha} = \frac{R}{\sin\gamma} \rightarrow ③$$

∴ From eqn (2) and (3), we get

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

---

Transmissibility of a force :-

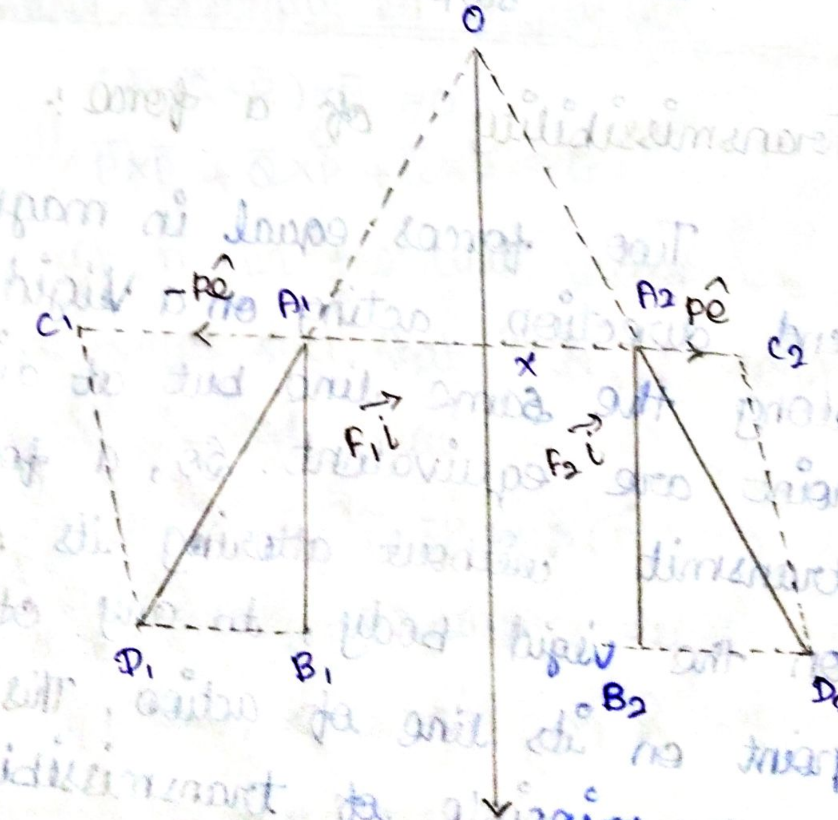
Two forces equal in magnitude and direction acting on a rigid body along the same line but at different point are equivalent. So, a force canbe transmit without altering its effect on the rigid body, to any other point on its line of action. This is known as the principle of transmissibility of force on a rigid body.

## Parallel forces :-

Forces whose lines of action are parallel are called Parallel forces. If their directions are in the same sense, then they are called like Parallel forces. Otherwise they are called Unlike Parallel forces.

To find the resultant of two Parallel forces acting on a rigid body.

case (i) :- Let the force be like Parallel forces namely $F_1\vec{i}$ and $F_2\vec{i}$ acting at $A_1$ and $A_2$ respectively where $\vec{i}$ is the unit vector in the direction of the forces.



Let $\hat{e}$ be the unit vector in the direction of $\overline{A_1A_2}$ introduce a force, $P\hat{e}$ at $A_1$ and a force $P\hat{e}$ at $A_2$. Since, These two forces are equal in magnitude and opposite in direction and act along the same line, their introduction will Not affect the effect of the given two forces. Let

$$\overline{A_1B_1} = F_1\vec{i} \quad A_2B_2 \quad \overline{A_2B_2} = F_2\vec{i} \quad \overline{A_1C_1} = -P\hat{e}$$

$$\overline{A_2C_2} = P\hat{e}$$

Complete the Parallelograms $A_1B_1D_1C_1$ and $A_2B_2D_2C_2$. Then the resultant of the two forces $F_1\vec{i}$ and $-P\hat{e}$ acting at $A_1$ is

$$\overline{A_1D_1} = F_1\vec{i} - P\hat{e}$$

and the resultant of the forces $F_2\vec{i}$ and $P\hat{e}$ acting at $A_2$ is

$$\overline{A_2D_2} = F_2\vec{i} + P\hat{e}$$

If the lin $A_1D_1$ and $A_2D_2$ intersent at resultants is

$$\overline{A_1D_1} + \overline{A_2D_2} = (F_1\vec{i} - P\hat{e}) + (F_2\vec{i} + P\hat{e})$$

$$= (F_1 + F_2)\vec{i}$$

acting at O, Note that this resultant is Parallel to the original forces.

Point of intersection of the resultant with $A_1, A_2$. From the similar triangles

$\Delta OXA_1$, $\Delta A_1B_1D$

$$\frac{OX}{XA_1} = \frac{F_1}{P}$$

Also from the similar triangles $\Delta OXA_2$

$\Delta A_2B_2D_2$

$$\frac{OX}{XA_2} = \frac{F_2}{P}$$

Dividing (2) by (1), we get

$$\frac{XA_1}{XA_2} = \frac{F_2}{F_1}$$

That is the line of action of a resultant divides interally $A_1A_2$ in the ratio $F_2 : F_1$

Note: In sums this result may be used in the form

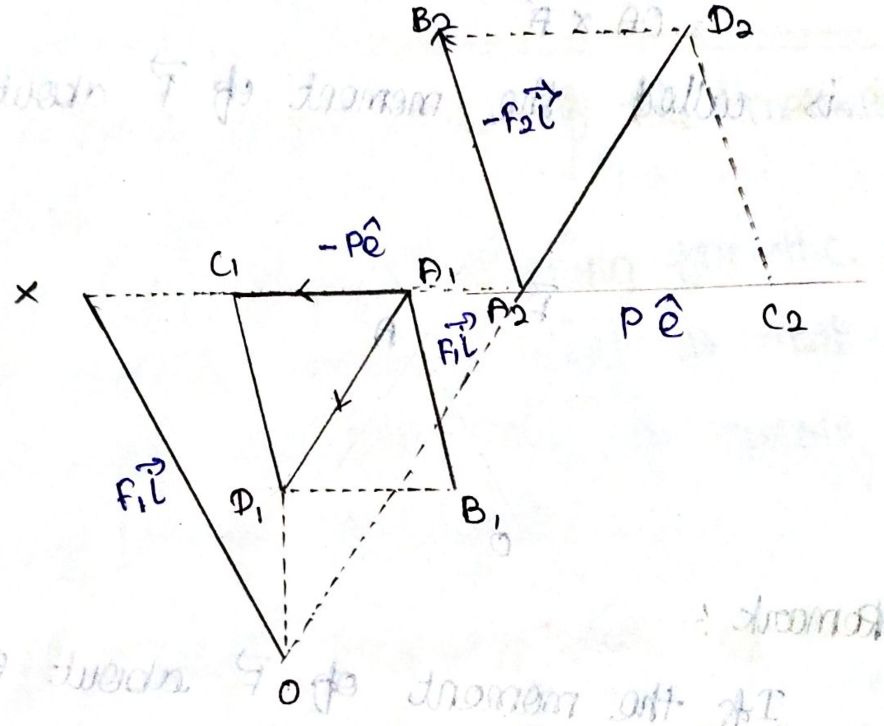
$$F_1 \times XA_1 = F_2 \times XA_2$$

---

**Position vector of X :-**

Let the Position vectors of $A_1, A_2$ be $\vec{r_1}, \vec{r_2}$ since X divides $A_1A_2$ interally in the ratio of $F_2 : F_1$. The position vector of X is $\dfrac{F_1\bar{r}_1 + F_2\bar{r}_2}{F_1+F_2}$.

case (ii) :- Let the given forces be unlike Parallel forces $F_1\vec{i}$ and $F_2(-\vec{i})$ $(F_1 > F_2)$ acting at $A_1$ and $A_2$ respectively. If we apply the produce followed in case (i) we see that the steps of case (i) respect with the only difference that instead of $F_2$ they have $-F_2$.




Then we get the resultant of the forces $F_1\vec{i}$ and $-F_2\vec{i}$ acting at $A_1$ and $A_2$ is

$$\{F_1 + (-F_2)\}\vec{i}$$

acting at the point which divides $A_1A_2$ in the ratio of $(-F_2) : F_1$ that is at the point which divides $A_1A_2$ externally in the ratio of $F_2 : F_1$

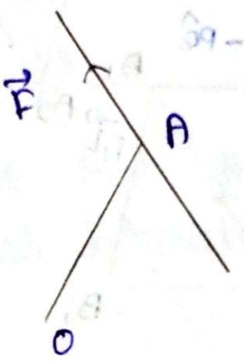
## Moment of a force:-

Let $\vec{F}$ be a force and A be a point on its line of action.

Let O be a point in space. Then the vector.

$$\overline{OA} \times \vec{F}$$

is called the moment of $\vec{F}$ about O.




Remark :-

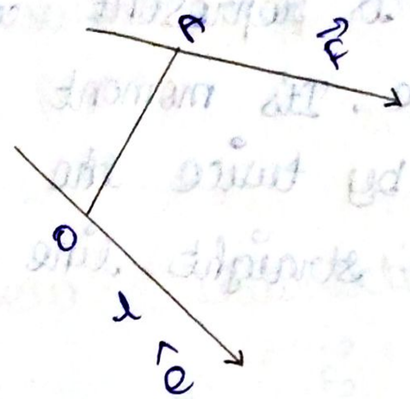
If the moment of $\vec{F}$ about A is zero either

(i) $\vec{F} = 0$ (or)

(ii) The line of action of $\vec{F}$ passes though A.

## Moment of a force about a line :-

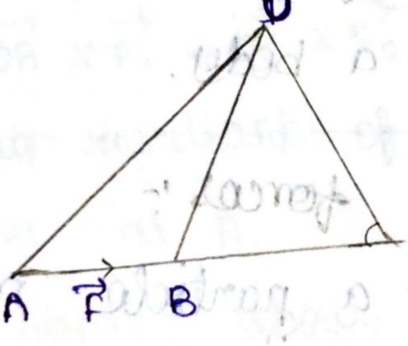
Let $\vec{F}$ be a force and A be a point on its line of action. Let $l$ be a directed line thoughts a point O. The direction of the line being specified by $\hat{e}$. Then the scalar triple product $(\vec{OA} \times \vec{F}).\hat{e}$ is called the moment of the force $\vec{F}$ about $l$.




Physical significare of the moment of a force :-

The physical meaning for the moment of a force about a point is that it measures the tendency to rotate the body about the point.

Geometrical representation of a moment :-




Let a force $\vec{F}$ acting a body represented in magnitude, direction and the line of acting by the line AB.

Let O be any given point and ON, the $\perp r$ from O on AB the moment of force F about O.

$= \vec{F} \times ON$

$= 2\left[\frac{1}{2} \times AB \times ON\right]$ = area $\triangle AOB$

If a force is represent completely by a straight line. It's moment about any point is given by twice the area of the triangle which straight line subtends at the Point.

Rigid Body :-

A system of particle is such that the distance any two of them is always constant is called Rigid body.

Applied forces :-

Force applied on a body by the external agences are called applied forces on a body.

Effective forces :-

If a particle mass m has an accoleration $\ddot{r}$ then a quantity m $\ddot{r}$ is call effective forces.

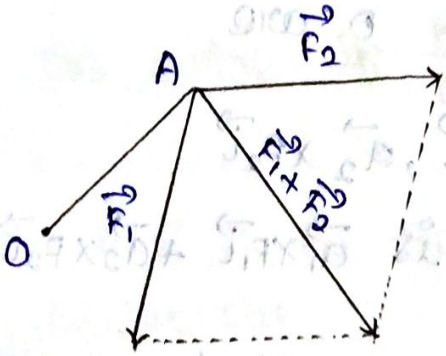
Varignon's Theorem :-

Statement :

(The sum of the moments of two intersecting or Parallel forces about any point is equal to the moment of the resultant of the forces about the same point).

Proof :-

Case(i) :- Intersecting forces




Let the lines of action of the forces $\vec{F_1}$ and $\vec{F_2}$ intersect at A. Then the moment of $\vec{F_1}$ and $\vec{F_2}$ about any point O are $\vec{OA} \times \vec{F_2}$ and their sum is

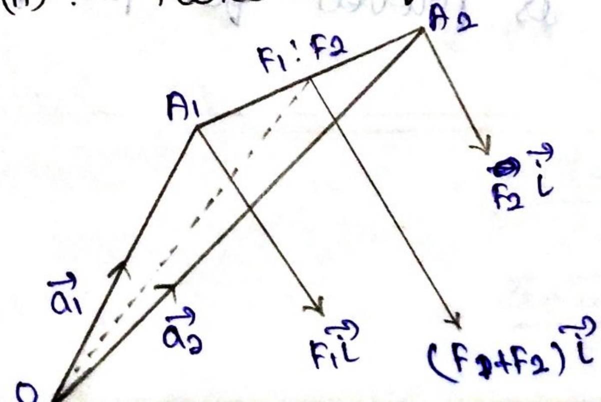
$$\vec{OA} \times \vec{F_1} + \vec{OA} \times \vec{F_2}$$

But the resultant of $\vec{F_1}$ and $\vec{F_2}$ is $\vec{F_1}+\vec{F_2}$ acting at A.

So its moment about O is $\vec{OA} \times (\vec{F_1} \times \vec{F_2})$

Since $\vec{OA} \times \vec{F_1} + \vec{OA} \times \vec{F_2} = \vec{OA} \times (\vec{F_1}+\vec{F_2})$, the theorem is proved for intersecting forces.

Case (ii) :- Parallel forces



Let the Parallel forces be $\vec{F_1} = F_1\vec{i}$ and $F_2 = F_2\vec{i}$ acting at $A_1$ and $A_2$ respectively. Let $\vec{a_1}, \vec{a_2}$ be the position vectors of with respect to O. Then the moment of $\vec{F_1}, \vec{F_2}$ about O are

$$\vec{a_1} \times F_1\vec{i}, \vec{a_2} \times F_2\vec{i}$$

Their sum is $\vec{a_1} \times F_1\vec{i} + \vec{a_2} \times F_2\vec{i} = (F_1\vec{a_1} + F_2\vec{a_2})\vec{i}$ → ①

But the resultant of $F_1\vec{i}$ and $F_2\vec{i}$ is $(F_1+F_2)\vec{i}$ acting at x where x divides $A_1A_2$ internally in the ratio $F_2 : F_1$

So, the Position vector of x is $\dfrac{F_1\vec{a_1} \times F_2\vec{a_2}}{F_1+F_2}$

So, the moment of the resultant about O is

$$\vec{OX} \times (F_1+F_2)\vec{i} = \frac{F_1\vec{a_1}+F_2\vec{a_2}}{F_1+F_2} \times (F_1+F_2)\vec{i}$$

$$= (F_1\vec{a_1} + F_2\vec{a_2})\vec{i} \rightarrow ②$$

∴ From eqn ① and ② the theorem is proved for parallel forces.

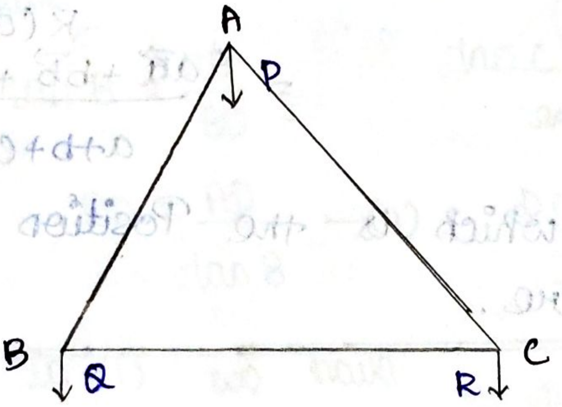
1. Three like Parallel forces P, Q, R act at the vertices of a triangle ABC so that the resultant passes through

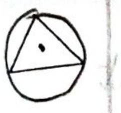
(i) The centroid if $P = Q = R$.

(ii) The centre if $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$

Soln:-

Let $\vec{a}, \vec{b}, \vec{c}$ be the Position vectors of A, B, C respectively





circumcentre

The resultant passes through the point whose position vector

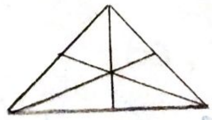
$$r = \frac{P\vec{a} + Q\vec{b} + R\vec{c}}{P+Q+R} \rightarrow (1)$$

(i) Given that $P = Q = R$ (say $= P$)

$\therefore$ eqn (1) becomes

$$\text{Position vector} = \frac{P\vec{a} + Q\vec{b} + R\vec{c}}{P+P+P}$$

$$= \frac{P\vec{a} + Q\vec{b} + R\vec{c}}{3P}$$


centroid

$$= \frac{P(\vec{a} + \vec{b} + \vec{c})}{3P}$$

$$= \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

which is the position vectors of the centroid.

(ii) Given that $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} = K$ (say)

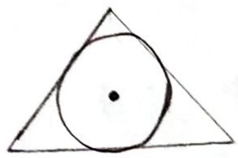
$P = aK$ ; $Q = bK$ ; $R = cK$

using this in eqn (1), we get

$$\text{Position vector} = \frac{aK\vec{a} + bK\vec{b} + cK\vec{c}}{K(a+b+c)}$$

$$= \frac{K(a\vec{a} + b\vec{b} + c\vec{c})}{K(a+b+c)}$$

$$= \frac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}$$

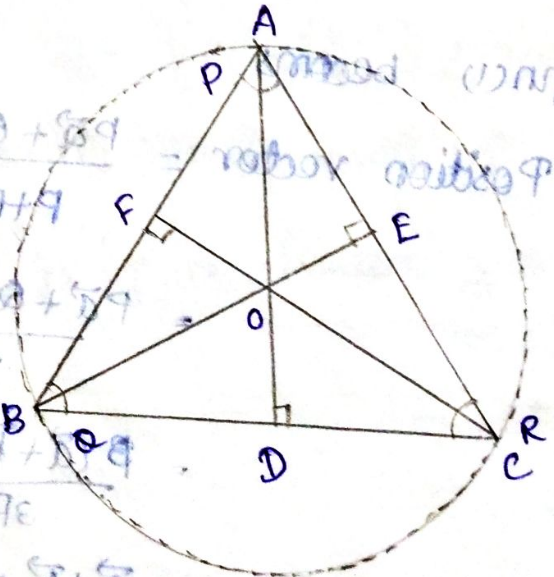

in centre

which is the Position vector of the incentre.

---

(2). Three like parallel forces P, Q, R act at the vertices of a triangle ABC. If their resultant passes through the orthocentre O. Show that $\frac{P}{\tan A} = \frac{Q}{\tan B} = \frac{R}{\tan C}$

Soln:-



Let AD be the altitudes, Though A

Now P act at A.

The resultant of Q, R should act at D such that

$$Q \times BD = DC \times R$$

$$\frac{BD}{DC} = \frac{R}{Q} \quad \rightarrow (1)$$

consider the triangle $\Delta ABD, \Delta ADC$

we have

$$\tan B = \frac{AD}{BD} \qquad \tan C = \frac{AD}{DC}$$

$$BD = \frac{AD}{\tan B} \rightarrow (2) \qquad DC = \frac{AD}{\tan C} \rightarrow (3)$$

(2), (3) in (1) we have

$$\frac{\cancel{AD}}{\tan B} \times \frac{\tan C}{\cancel{AD}} = \frac{R}{Q}$$

$$\frac{\tan C}{\tan B} = \frac{R}{Q}$$

$$\frac{Q}{\tan B} = \frac{R}{\tan C} \rightarrow (4)$$

Let BE be the altitudes, they B, Now Q act at B.

So, the resultant of R, P should act at E such that

$$R \times CE = EA \times P$$

$$\frac{CE}{EA} = \frac{P}{R} \rightarrow (5)$$

consider the triangle $\Delta BCE$, $\Delta BAE$
we have

$$\tan C = \frac{BE}{CE} \Rightarrow CE = \frac{BE}{\tan C}$$

$$\tan A = \frac{BE}{EA} \Rightarrow EA = \frac{BE}{\tan A}$$

(5) Becomes

$$\frac{BE}{\tan C} \times \frac{\tan A}{BE} = \frac{P}{R}$$

$$\frac{\tan A}{\tan C} = \frac{P}{R} \Rightarrow \frac{P}{\tan A} = \frac{R}{\tan C} \rightarrow (6)$$

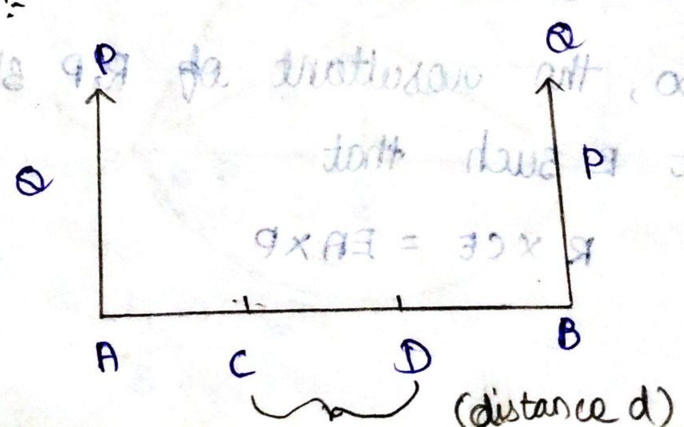
from (4), (6)

$$\frac{P}{\tan A} = \frac{Q}{\tan B} = \frac{R}{\tan C}$$

---

3). If two like parallel forces of magnitude $P, Q$ $(P > Q)$ acting on a rigid body at $A, B$ are inter changed in Position show that the line of action of the resultant is displaced through a distance $\frac{AB(P-Q)}{P+Q}$.

solns :-



Let P, Q be two like Parallel forces act at A, B

Let 'C' be the resultant of two two Parallel forces. Then

$$\frac{AC}{CB} = \frac{Q}{P}$$

$$P.AC = Q.CB \rightarrow (1)$$

When the forces P, Q are interchanged then D is the new resultant, we have

$$\frac{AD}{DB} = \frac{P}{Q}$$

$$Q.AD = P.DB \longrightarrow (2)$$

Let distance between 'C' and 'D' is 'd'.

from (2)

$$Q.(AC+CD) = P.(CB-CD)$$

$$Q.AC + Q.CD = P.CB - P.CD$$

$$Q.AC + Q.d = P.CB - P.d$$

$$P.d + Q.d = P.CB - Q.AC$$

$$(P+Q).d = P.CB - Q.AC$$

$$= P(AB-AC) - Q(AB-CB)$$

$$= PAB - PAC - QAB + QCB$$

$$= PAB - \cancel{QCB} - QAB + \cancel{QCB} \text{ [Using (1)]}$$

$$(P+Q)\cdot d = (P-Q)\,AB$$

$$d = \frac{AB\,(P-Q)}{(P+Q)}$$ lt.

4). ABC is a given triangle. Forces P, Q, R acting along the lines OA, OB, OC are in equilibrium. Prove that

(i) $P:Q:R = a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)$

if 'O' is the circumcentre of the ABC (triangle).

(ii). $P:Q:R = \cos A/2 \cdot \cos B/2 \cdot \cos C/2$ is 'O' is the incentre of the $\triangle ABC$.

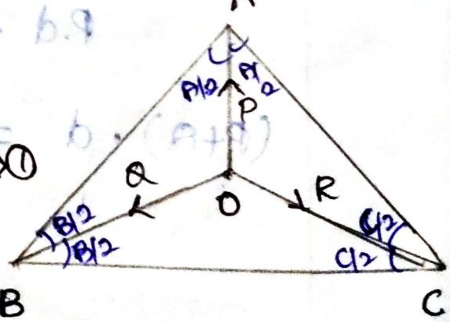
(iii). $P:Q:R = a:b:c$ if 'O' is the ortho-centre of the $\triangle ABC$

(iv) $P:Q:R = OA:OB:OC$ if 'O' is the centroid of the $\triangle ABC$.

Soln :-

(i). By Lami's theorem,

$$\frac{P}{\sin\angle BOC} = \frac{Q}{\sin\angle COA} = \frac{R}{\sin\angle AOB} \to (1)$$




When 'O' is the circumstance of the triangle $\triangle ABC$, we've

$\angle BOC = 2A$ ; $\angle COA = 2B$, $\angle AOB = 2C$

(1) becomes :-

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

* $\sin 2A = 2\sin A \cos B$
* $\Delta ABC = \frac{1}{2} bc \sin A$
* $\Delta ABC = \frac{1}{2} ca \sin B$
* $\Delta ABC = \frac{1}{2} ab \sin C$

$$\frac{P}{2\sin A\cos A} = \frac{Q}{2\sin B\cos B} = \frac{R}{2\sin C\cos C} \rightarrow (2)$$

We know that

Area of the triangle பரப்பு

$\Delta ABC = \frac{1}{2} bc \sin A$ $\therefore \sin A = \frac{2\Delta ABC}{bc}$

Similiarly $\sin B = \frac{2\Delta ABC}{ca}$ ; $\therefore \sin C = \frac{2\Delta ABC}{ab}$

$\cos A = \frac{b^2+c^2-a^2}{2bc}$ ; $\cos B = \frac{c^2+a^2-b^2}{2ca}$

$\cos C = \frac{a^2+b^2-c^2}{2ab}$

$\therefore$ equation (2) becomes

$$\frac{P}{4\frac{\Delta ABC}{bc}\left(\frac{b^2+c^2-a^2}{2bc}\right)} = \frac{Q}{4\frac{\Delta ABC}{ca}\left(\frac{c^2+a^2-b^2}{2ca}\right)} = \frac{R}{4\frac{\Delta ABC}{ab}\left(\frac{a^2+b^2-c^2}{2ab}\right)}$$

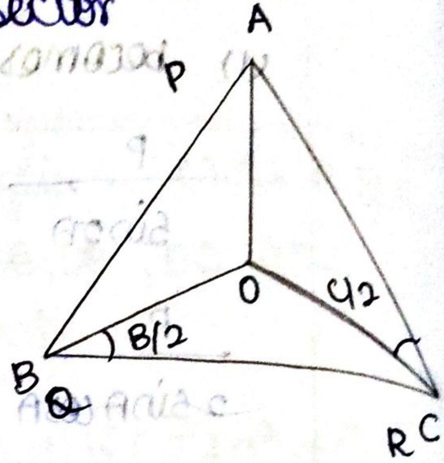
$$\frac{P b^2 c^2}{2\Delta ABC(b^2+c^2-a^2)} = \frac{Q c^2 a^2}{2\Delta ABC(c^2+a^2-b^2)} = \frac{R a^2 b^2}{2\Delta ABC(a^2+b^2-c^2)}$$

Multiplying throughout by $2\frac{\Delta ABC}{a^2b^2c^2}$, we got

$$\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}$$

$\therefore P : Q : R = a^2(b^2+c^2-a) : b^2(c^2+a^2-b^2)$
$: c^2(a^2+b^2-c^2)$

(ii) When 'O' is the incentre of $\Delta ABC$, we have OB, OC are the bisector of the angles $\angle B$ and $\angle C$




$$\angle BOC + \angle B/2 + \angle C/2 = 180^\circ$$

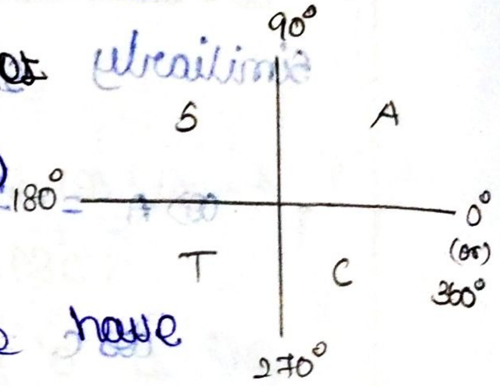
$$\angle BOC = 180^\circ - B/2 - C/2$$

$$= 180^\circ - \left(\frac{B+C}{2}\right) \rightarrow (1)$$

But $A+B+C = 180^\circ \Rightarrow \frac{B+C}{2} = \frac{180^\circ - A}{2}$

$$\frac{B+C}{2} = 90 - A/2 \rightarrow (2)$$

Using (2) and (1), we get




$$\angle BOC = 180^\circ - (90^\circ - A/2)$$

$$\angle BOC = 90^\circ + A/2$$

By Lami's theorem, we have

$$\frac{P}{\sin \angle BOC} = \frac{Q}{\sin \angle COA} = \frac{R}{\sin \angle AOB} \rightarrow (3)$$

$\sin \angle BOC = \sin(90^\circ + A/2) = \cos A/2$

$|||^{ly}$

$\sin \angle COA = \cos B/2$

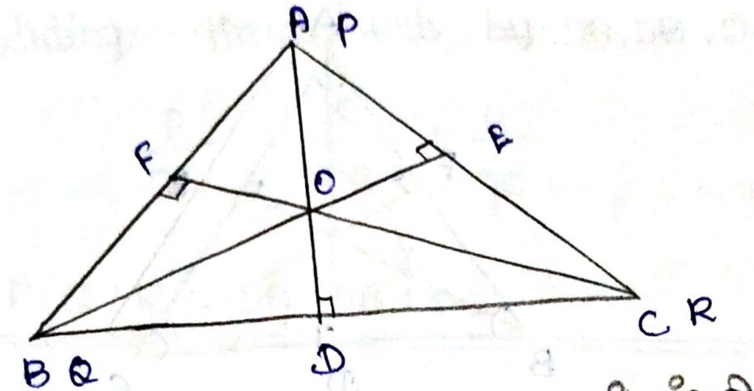
$\sin \angle AOB = \cos C/2$

eqn (3) becomes,

$$\frac{P}{\cos A/2} = \frac{Q}{\cos B/2} = \frac{R}{\cos C/2}$$

$\therefore P : Q : R = \cos A/2 : \cos B/2 : \cos C/2$

iii).




When 'O' is the orthocentre of $\triangle ABC$, we have

AD, BE, CF are the altitude Quadrilateral AFOE is

$\angle AFO + \angle OEA = 90^\circ + 90^\circ = 180^\circ$

$\angle FOE + \angle A = 180^\circ$

$\angle FOE = 180^\circ - A$

$\angle BOC = \angle FOE$ (Vertically opposide angles).

$= 180^\circ - A$

$\sin \angle BOC = \sin(180^\circ - A) = \sin A$

III$^{ly}$ $\sin \angle COA = \sin B$

$\sin \angle AOB = \sin C$

By Lami's theorem, we have

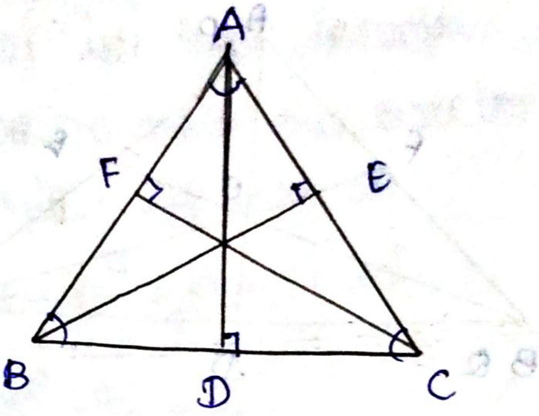
$$\frac{P}{\sin \angle BOC} = \frac{Q}{\sin \angle COB} = \frac{R}{\sin \angle AOB}$$

ie $\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$

By sine Property,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow P : Q : R = a : b : c$

H.




when 'o' is the centroid of $\Delta ABC$, we have.

AD, CF are the medians to sine property, we have

$\Delta BOC = \Delta COA = \Delta AOB = \frac{1}{3} \Delta ABC$

$\Delta BOC = \frac{1}{2} OB.OC . \sin \angle BOC$

$\frac{1}{3} \Delta ABC = \frac{1}{2} OB . OC \sin \angle BOC$

$\sin \angle BOC = \frac{2}{3} \cdot \frac{\Delta ABC}{OB.OC}$

III^ly

$\sin \angle COA = \frac{2}{3} \frac{\Delta ABC}{OC.OA}$

$\sin \angle AOB = \frac{2}{3} \frac{\Delta ABC}{OA.OB}$

By Lami's theorem, we have

$$\frac{P}{\sin \angle BOC} = \frac{Q}{\sin \angle COA} = \frac{R}{\sin \angle AOB}$$

$$\frac{P}{\frac{2}{3} \frac{\Delta ABC}{OB.OC}} = \frac{Q}{\frac{2}{3} \frac{\Delta ABC}{OC.OA}} = \frac{R}{\frac{2}{3} \frac{\Delta ABC}{OA.OB}}$$

$P.OB.OC = Q.OC.OA = R.OA.OB.$

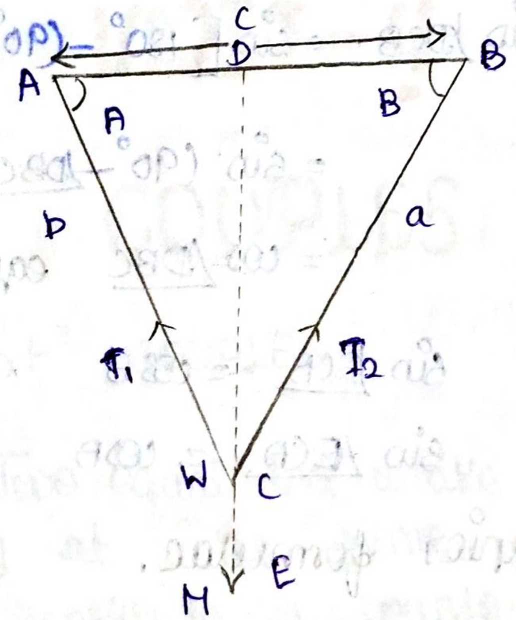
Dividing throughout by OA, OB, OC, we get

$$\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{OC}$$

$$P : Q : R = OA : OB : OC$$

5). A and B are two fixed points on a horizontal at distance small c a parts. Two light strings are AC and BC of lengths b and a respy. Support mass etc... Show that the tensions of the strings are ratio $b(a^2+c^2-b^2) : a(b^2+c^2-a^2)$

Soln :-




Let $T_1$ and $T_2$ be the tension along the strings CA and CB.

Let 'W' be the weight of the mass at 'C' acting vertically downwards along CE.

Produce EC to meet AB at D.

since D is at rest under the action of three forces namely

(i) $T_1$ acts along CA

(ii) $T_2$ acts along CB

(iii) W acts along CE

From Lami's theorem, we have

$$\frac{T_1}{\sin \angle ECB} = \frac{T_2}{\sin \angle ECA} \rightarrow (1) \quad \angle ECB = 180^\circ - \angle DCB$$

$$\sin \angle ECB = \sin[180^\circ - \angle DCB]$$

$$\sin \angle ECB = \sin \angle DCB \rightarrow (2)$$

$$\sin \angle DCB = \sin[180^\circ - (90^\circ + \angle DBC)]$$

$$= \sin(90^\circ - \angle DBC)$$

$$= \cos \angle DBC$$

equ(2) becomes, we get

III$^{ly}$

$$\left.\begin{aligned} \sin \angle ECB &= \cos B \\ \sin \angle ECA &= \cos A \end{aligned}\right\} \rightarrow (3)$$

By Napier formulae.

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{a^2+c^2-b^2}{2ac} \rightarrow (4)$$

From (1),

$$\frac{T_1}{\cos B} = \frac{T_2}{\cos A} \quad \text{using (3)}$$

Put (A) in eqn (3), we got

$$\frac{T_1}{\frac{a^2+c^2-b^2}{2ac}} = \frac{T_2}{\frac{b^2+c^2-a^2}{2bc}}$$

$$\frac{T_1}{\frac{a^2+c^2-b^2}{a}} = \frac{T_2}{\frac{b^2+c^2-a^2}{b}}$$

$T_1, T_2$ are in the ratio $\frac{a^2+c^2-b^2}{a} ; \frac{b^2+c^2-a^2}{b}$

Multiply in ab,

$$T_1 : T_2 = b(a^2+c^2-b^2) : a(b^2+c^2-a^2)$$