INDEX

Title	Page No.	Teacher's Sign / Remarks

Real Analysis:

Real analysis or theory of functions of a real variable is a branch of Mathematical analysis dealing with the set of real numbers. In particular, it deals with the analytic properties of real functions and sequences, including convergence and limits of sequences of real numbers, the calculus of the real numbers, and continuity, smoothness and related properties of real valued functions.

Scope: Real analysis is an area of analysis which studies concepts such as sequences and their limits, continuity, differentiation, integration and sequences of functions. By definition, real analysis focuses on the real numbers, often including positive or negative infinity.

Introduction:

The union of rational and irrational numbers constitute the real number system.

Rational Numbers:

Quotients of integers a/b (where $b \neq 0$) are called rational numbers.

Ex: $1/2$, $-7/5$, 6.

Irrational Numbers:-

Real nos that are not rational are called irrational.

Ex: $\sqrt{2}$, e, π, e^{π}

Results:

* If a and b are real numbers with $a<b$ then there is a rational number x and an irrational number y such that $a<x<b$, $a<y<b$ (ie) between any two distinct real numbers there is a rational and an irrational.
* If a and b are real numbers, then the geometric interpretation of $|a-b|$ is the distance from a to b. (or b to a).

Defns

* Set: Collection of well defined objects. The objects in the set are called elements or points.

Union: $A \cup B$ is the set of all elements in either A or B or both. If A, B are sets

$$A \cup B = \{x / x \in A \text{ or } x \in B\}$$

Intersection:

If A and B are sets, then $A \cap B$ is the set of all elements in both A and B

$$A \cap B = \{x / x \in A \text{ and } x \in B\}$$

Empty Set: The set which has no elements.

Equal set: We say that two sets are equal if they contain precisely the same elements

Axioms:

The Real numbers R satisfy the ten axioms. The axioms fall into three groups such as field axioms, order axioms and the completeness axiom (also called l.u.b axiom or the axiom of the continuity).

Field Axioms: If $x, y, z \in R$

1. $x+y = y+x$, $xy = yx$ (Commutative)

2. $x+(y+z) = (x+y)+z$, $x(yz) = (xy)z$ (associative).

3. $x(y+z) = xy+xz$ (distributive)

4. $x+z=y$, $z=y-x$, $x-x=0$, $-x$ is the negative of x.

5. There exist at least one real number $x \neq 0$ then $xz=y$, $z=y/x$. x^{-1} or $1/x$ if $x \neq 0$ we call x^{-1} is the reciprocal of x.

The order axioms:

6. Exactly one of the relations $x=y$, $x<y$, $x>y$ holds.

7. If $x<y$ then for every z we have $x+z < y+z$

8. If $x>0$ and $y>0$ then $xy>0$

9. If $x>y$ and $y>z$ then $x>z$.

Completeness axiom:

10. Every non-empty set S of real numbers which is bounded above

has a supremum (ie) there is a real numbers b such that $b = \sup S$.

Results:

* If x and y are real numbers and if $x<y$ then $-x>-y$. Also if $x \neq 0$, $y \neq 0$ and $x<y$ then $1/x > 1/y$.
* For $x>0$ we define $|x|$ to be x. For $x<0$ we define $|x|$ to be $-x$.
* $|x+y| \leq |x|+|y|$ and $|xy| = |x||y|$
* If a, b, c are real numbers then $|a-b| \leq |a-c|+|c-b|$.

Functions:

If A and B are sets, then the cartesian product of A and B (denoted by $A \times B$) is the set of all ordered pairs $\langle a,b \rangle$ where $a \in A$ and $b \in B$.

The ordered triple (A, B, f) where A is the domain, B is the co-domain and f is the set of ordered pairs.

The relation between domain and codomain is called functions.

Mapping: Let A and B be any two sets. A function f from A into B is a subset $A \times B$ with the property that each $a \in A$ belongs to precisely one pair (a, b).
Instead of $(x, y) \in f$ we usually write $y = f(x)$. Then y is called the image of x under f. The set A is called the domain of f. The range of f is the set $\{b \in B / b = f(a)$ for some $a\}$

i.e., the range of f is the subset of B consisting of all images of elements of A. Such a function is called a mapping of A into B.

One-to-One

If $f : A \to B$ then f is one-to-one (denoted as 1-1) if

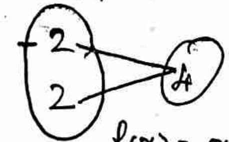
$$\forall\, x, y \in A \quad f(x) = f(y) \Rightarrow x = y \text{ (or)}$$

$$\forall\, x, y \in A \quad x \neq y \text{ implies } f(x) \neq f(y)$$

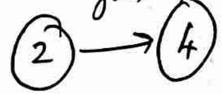
Example:

A function f defined by $f(x) = x^2$ $(-\infty < x < \infty)$ is not 1-1 but the function g defined by $g(x) = x^2$ $(0 \le x \le \infty)$ is 1-1.

(eg) In $f(x) = x^2$ $(-\infty < x < \infty)$, 4 is the image of both +2 and −2, but for $g(x) = x^2$ $(0 \le x \le \infty)$, 4 is the image of 2 hence 1-1



$f(x) = x^2$

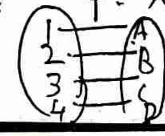
$g(x) = x^2$


Note: Elements of the co-domain are being mapped to by atmost one element of the domain not every element of the co-domain.

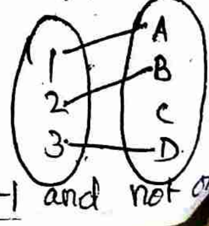
Onto: A function is onto if every element of the codomain is mapped to by some element of the domain.

If $f : A \to B$ then $\forall\, y \in B$, $\exists\, x \in A$ such that $y = f(x)$.

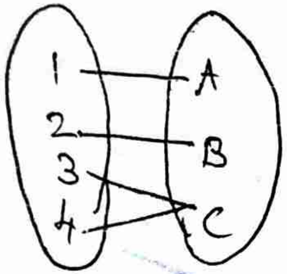
Note: $f : X \to Y$



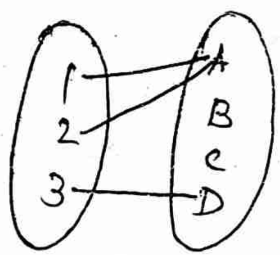
1-1 and onto

X Y


1-1 and not onto



not 1-1 and onto



Not 1-1 and not onto.

Inverse function:

If $f: A \to B$ and f is 1-1, then the function f^{-1} (called the inverse of f) is defined as follows:

If $f(a) = b$ then $f^{-1}(b) = a$.

Thus the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

Equivalent:

If $f: A \xRightarrow{\text{onto}} B$ and f is 1-1, then f is called a 1-1 correspondence between A and B. If there exists a 1-1 correspondance between the sets A and B, then A and B are called equivalent.

Note: 1. Every set A is equivalent to itself.

2. If A and B are equivalent, then B and A are equivalent.
3. If A and B are equivalent and B and C are equivalent, then A and C are equivalent.

Example:

1. Set of all integers and the set of all rational numbers are equivalent
   $f: Z \to Q$.
2. Set of all integers and the set of all real numbers are not equivalent
   $f: Z \to R$

Infinite Set:

The set A is said to be infinite if, for each positive integer n, A contains a subset with precisely n elements.

eg: If n is a positive integer then the set of all +ve integers I, $I = \{1, 2, \ldots, \infty\}$

is an infinite set. Because it has a proper subset $2^n = \{2, 4, 8, \dots\}$ for each $n \in I$ (eg) set of Real nos R.

Finite set

A set that is not infinite is called a finite.

Countable and uncountable :

A set A is said to be countable (or denumerable) if A is equivalent to the set I of positive integers. An uncountable set is an infinite set which is not countable.

The set A is countable if there exist a 1-1 ~~function~~ correspondence between I and A ie $f : I \xrightarrow{\text{onto}} A$. The elements of A are then the images $f(1), f(2), \dots$ of positive integers.

(ie) $A = \{f(1), f(2), \dots\}$ where all are distinct from one another.

Example:

The set of all integers is countable. For by arranging the integers as $0, -1, +1, -2, +2, \dots$ we give a scheme by which they can be counted.

The function f defined by

$$f(n) = \frac{n-1}{2}, \quad n = 1, 3, 5, \dots$$

$$f(n) = -n/2, \quad n = 2, 4, 6, \dots$$

is a 1-1 correspondence between I and the set of all integers. For $f(1), f(2), \dots$ is the same as $\{0, -1, 1, -2, 2, \dots\}$.

Note:

A set is countable means that its elements can be counted (arranged with labels $1, 2, 3, \dots$) we usually write $a_1, a_2, \dots$ instead of $f(1), f(2), \dots$

Example:

If A and B are countable then so is $A \cup B$

If $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$

Thus $a_1, b_1, a_2, b_2, \dots$ is a scheme for counting the elements $A \cup B$.

Theorem 1.1

If $A_1, A_2, \ldots$ are countable sets then $\bigcup_{n=1}^{\infty} A_n$ is countable. In words the, countable union of countable sets is countable.

Proof: We may write: $A_1 = \{a_1^1, a_2^1, \ldots\}$
$A_2 = \{a_1^2, a_2^2, \ldots\} \;\ldots\; A_n = \{a_1^n, a_2^n, \ldots\}$
so that a_k^j is the k^{th} element of the set A_j.

Define the height of a_k^j to be $j+k$.
Then a_1^1 is the only element of height 2.
Likewise a_2^1 and a_1^2 are the only elements height 3 and so on.

We may arrange the elements $\bigcup_{n=1}^{\infty} A_n$ according to their heights as

$$a_1^1, a_1^2, a_2^1, a_3^1, a_2^2 a_1^3, a_1^4, \ldots$$

Any a_k^j once counted should not be repeated.

Pictorially, we are listing the elements $\bigcup_{n=1}^{\infty} A_n$ in the following array and counting them in order are indicated by arrows.

$$\begin{array}{ccccc}
a_1^1 & a_2^1 \rightarrow & a_3^1 & a_4^1 & \ldots \\
\downarrow \nearrow & \swarrow & \nearrow & & \\
a_1^2 & a_2^2 & a_3^2 & a_4^2 & \ldots \\
a_1^3 & a_2^3 & a_3^3 & a_4^3 & \ldots \\
\downarrow \nearrow & & & & \\
a_1^4 & a_2^4 & a_3^4 & a_4^4 & \ldots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

This counting scheme counts every a_k^j proves that $\bigcup_{n=1}^{\infty} A_n$ is countable.

Corollary: The set of all rational numbers is countable.

Proof: The set of all rational numbers is the

The union $\bigcup_{n=1}^{\infty} E_n$ where E_n is the set of rationals which can be written with denominator n. (ie) $E_n = \{0/n, -1/n, 1/n, -2/n, 2/n, \ldots\}$

Now each E_n is clearly equivalent to the set of all integers.

$\therefore$ E_n is countable.

Hence the set of all rationals is the countable union of countable sets.

Since countable union of countable sets is also countable, set of all rational also countable

Theorem: 1.2 If B is an infinite subset of the countable set A, then B is countable.

Proof:

Let $A = \{a_1, a_2, \ldots\}$

Then each element of B is an a_i.

Let n_1 be the smallest subscript for which $a_{n_1} \in B$, let n_2 be the next smallest and so on. Then $B = \{a_{n_1}, a_{n_2}, \ldots\}$.

The elements of B are thus labeled with $1, 2, \ldots$ and so B is countable.

Corollary: The set of all rational numbers $[0,1]$ is countable.

Since the set of all rational number is countable. and $[0,1]$ is an infinite subset of the countable set, set of rational numbers.

And hence $[0,1]$ is countable.

Exercise: => Exercise

① Is $f(x) = x^2$, $x \in R$ is one-to-one?

Soln:

The given function is not 1-1.

because $a \neq -a$, but $f(a) = f(-a) = a^2$.

(2) Is $f(x) = x$, $x \in \mathbb{R}$ a one-to-one fun?

the given function is 1-1.

$\therefore a \neq -a$, also $f(a) \neq f(-a)$.

(3) Is $f(x) = ax+b$, $x \in \mathbb{R}$, $a, b \in \mathbb{R}$ a 1-1 fun?

To show $f(x)$ to be a 1-1 fun

we should prove if $f(x_1) = f(x_2)$ then

$x_1 = x_2$.

$$ax_1 + b = ax_2 + b$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is 1-1.

(4) Is $f(x) = e^x \ (-\infty < x < \infty)$ a 1-1 fun?

let $f(x_1) = f(x_2)$

$$e^{x_1} = e^{x_2} \Rightarrow e^{x_1 - x_2} = 1$$

Taking log $x_1 - x_2 = 0$

$\therefore x_1 = x_2$

$\therefore$ It is 1-1.

5. Is $f(x) = e^{-x^2}$, $-\infty < x < \infty$ a 1-1 fun?

W.K.T $a \neq -a$ but $f(a) = f(-a) = e^{a^2}$.

$\therefore f$ is 1-1.

Theorem: 1.3

The set $[0,1] = \{x / 0 \leq x \leq 1\}$ is uncountable.

Proof:

Suppose that $[0,1]$ were countable.

Then $[0,1] = \{x_1, x_2, \ldots\}$ where every number in $[0,1]$ occurs among the x_i.

Expanding each x_i in decimals we have

$$x_1 = 0 \cdot a_1^1 a_2^1 a_3^1 \ldots$$

$$x_2 = 0 \cdot a_1^2 a_2^2 a_3^2 \ldots$$

$$\vdots$$

$$x_n = 0 \cdot a_1^n a_2^n \ldots a_n^n \ldots$$

Here a_i^j is the i^{th} element in x_j and $a_i^j \in \{0, 1, 2, \ldots 9\}$

let b_i be any integer from 0 to 8. Such that $b_1 \neq a_1^1$. Then let b_2 be any integer

from 0 to 8 such that $b_2 \neq a_2^2$.
In general, for each $n = 1, 2, \ldots$ let b_n be any integer from 0 to 8 such that $b_n \neq a_n^n$. Let $y = 0 \cdot b_1 b_2 \ldots b_n \ldots$ Then for any n, the decimal expansion for y differs from the decimal expansion for x_n since $b_n \neq a_n^n$. Moreover the decimal expansion for y is unique since no b_n is equal to 9. Hence $y \neq x_n$ for every n and $0 \leq y \leq 1$ which contradicts the assumption that every number in $[0,1]$ occurs among the x_i. This contradiction proves the theorem.

Corollary: The set R of all real numbers is uncountable.

Proof: Suppose the let the set R be a countable set. We know that $[0,1]$ is a subset of R. But the set $[0,1]$ is an uncountable set. The subset of a countable set is an uncountable set. Therefore it is a contradiction. Hence R is an uncountable set.

Binary Expansion:

The expansion of a real no. x in $[0,1]$ using the digits 0, 1 is called binary expansion.

For example $x = 0 \cdot a_1 a_2 a_3 \ldots$ means

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots$$

so that
$$1/2 = 0 \cdot 1000 \ldots \ (2)$$
$$1/4 = 0 \cdot 01000 \ldots \ (2)$$
$$1/16 = 0 \cdot 00010 \ldots \ (2)$$
$$1/2 + 1/4 + 1/16 = 0 \cdot 110100 \ldots \ (2)$$

where (2) denotes binary expansion.

Ternary expansion:

The expansion of a real no. x in $[0,1]$ using the digits 0, 1, 2 is called a ternary expansion.

For example. $x = 0.b_1 b_2 b_3 \ldots\ (3)$

means $x = \frac{b_1}{3} + \frac{b_2}{3^2} + \frac{b_3}{3^3} + \ldots$

so that

$$0.1012 = \frac{1}{3} + \frac{0}{3^2} + \frac{1}{3^3} + \frac{2}{3^4} + \ldots\ (3)$$

$$0.012 = \frac{0}{3} + \frac{1}{3^2} + \frac{2}{3^3} + \ldots\ (3)$$

Cantor set K:

The cantor set K is the set of all numbers x in $[0,1]$ which have a ternary expansion without the digit 1.

Example:

$$1/3 = 0.0222 \ldots\ (3)$$

and $2/3 = 0.2000 \ldots$

are in K but any x such that $1/3 < x < 2/3$ is not in K.

Note:

Bounded above and Bounded below:

The subset $A \subset R$ is said to be bounded above if there is a number $N \in R$ such that $x \le N$ for every $x \in A$. Here N is called the upper bound. The subset $A \subset R$ is said to be bounded below if there exist a number $M \in R$ such that $M \le x$ for every $x \in A$. Here M is called the lower bound. If A is both bounded below and bounded above we say that A is bounded.

Ex:

1) Consider the set of all negative integers, 0 is the upper bound of this set and it is bounded above. $[0,1]$ is bounded.

2) Consider the set of all +ve integers $A = \{1, 2, 3, \ldots \infty\}$, 1 is the lower bound of this set A and is bounded below.

Least upper bound (Supremum)

Let the subset $A \subset R$ be bounded above. The number L is called the least upper bound (l.u.b) for A if

① L is an upper bound for A and

② No number smaller than L is an upper bound for A.

Greatest lower bound (Infimum)

Let the subset $A \subset R$ be bounded below. The number l is called the greatest lower bound (g.l.b) for A if

① l is a lower bound for A and

② No number greater than l is a lower bound for A.

Examples:

1) The set $R^+ = (0, \infty)$ is unbounded above. It has no upper bounds and

no maximum element. It is bounded below by 0 but has no minimum element.

2) The closed interval $S = [0,1]$ is bounded above by 1 and bounded below by 0. The maximum element is 1 and min $S =$

3) The half open interval $S = [0,1)$ is bounded above by 1, but it has no maximum element. Its minimum element is 0. Here the number 1 is a least upper bound for S even though S has no maximum element.

Least upper bound axiom:

If A is any non-empty subset of R that is bounded above then A has a least upper bound in R.

eg: $A \subset R$, $A = \{1, 1.4, 1.41, 1.414, \dots\}$

the lub for A is $\sqrt{2}$.

Theorem: 1.4

If A is any non-empty subset of R that is bounded below, then A has a greatest lower bound in R.

$A = \{1, 2, 3, 4, 5, 6 \ldots\}$

$B = \{-1, -2, -3, -4 \ldots\}$

proof: let $B \subset R$ be the set of all $x \in R$. such that $B = \{-x / x \in A\}$

$B = \{-1 / 1 \in A\}$

(ie) The elements of B are the negatives of the elements of A. If M is the lower bound for A, then $-M$ is an upper bound for B. (Since $M \leq x$ for B $M \leq -x$, $x \leq -M$)

Hence B is bounded above so that by l.u.b axiom, B has a l.u.b.

If Q is the l.u.b for B then $-Q$ is the g.l.b for A.

$\therefore$ A has greatest lower bound in R.

Examples:

① If A and B are countable sets then show that $A \times B$ is a countable set.

soln: let A and B be countable sets.

Then $A = \{a_1, a_2, a_3, \ldots\}$ and

$B = \{b_1, b_2, b_3, \ldots\}$

Then $A \times B$ is written as

$A_1 = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots\}$

$A_2 = \{(a_2, b_1), (a_2, b_2), (a_2, b_3), \ldots\}$

$\vdots$

$A_i = \{(a_i, b_1), (a_i, b_2), (a_i, b_3), \ldots\}$

Here we know that $A_1, A_2, A_3, \ldots$ are countable sets.

$\therefore$ $A_1 \cup A_2 \cup A_3 \ldots$ is also countable.

$\therefore$ $A \times B$ is a countable set.

(2) S.T $Z \times Z$ is a countable set.

Soln: We know that the set of integers is a countable set.

$\therefore Z = \{a_1, a_2, a_3, \dots\}$

$A_1 = \{(a_1,a_1), (a_1,a_2), (a_1,a_3), \dots\}$

$A_2 = \{(a_2,a_1), (a_2,a_2), (a_2,a_3), \dots\}$

$\vdots$

$A_i = \{(a_i,a_1), (a_i,a_2) (a_i,a_3), \dots\}$

$\vdots$

Here $A_1, A_2, \dots$ are countable sets.

$\therefore A_1 \cup A_2 \cup A_3 \dots$ are also countable.

$\therefore Z \times Z$ is a countable set

Note: Let $A \subset R$. If the l.u.b and g.l.b of A are equal what can you say about the set?

Since the set A has the same number as g.l.b and l.u.b, the set has only one element.

Sequences of real numbers:

A sequence $S = \{S_i\}_{i=1}^{\infty}$ of real numbers is a function from I into R.

The real number S_i is $S(i)$. Instead of S as $\{S_i\}_{i=1}^{\infty}$ we sometimes write $S_1, S_2, \dots$. The numbers S_i $(i = 1, 2, \dots)$ is called the i^{th} term of the sequence.

Subsequence:

If $S = \{S_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $N = \{n_i\}_{i=1}^{\infty}$ is a subsequence of the sequence of positive integers, then the composite function $S \circ N$ is called the subsequence of S.

For $i \in I$ we have $N(i) = n_i$

$S \circ N(i) = S[N(i)] = S(n_i) = S_{n_i}$

and hence So $N = \{S_{n_i}\}_{n=1}^{\infty}$.

Examples:

1) The sequence of primes 2, 3, 5, 7, 11, ...
2) The set of even nos 2, 4, 6, 8, ...
3) The set of odd nos 1, 3, 5, ...

Hint: (i) $N = 1, 2, 3, 4, 5, \ldots$ without changing the order if any subset can be taken out from N then that is called subsequence.

Limit of a Sequence:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that S_n approaches the limit L (as n approaches infinity) if for every $\epsilon > 0$, there is a positive integer N such that $|S_n - L| < \epsilon \ (n \geq N)$ $\longrightarrow$ (1)

If S_n approaches the limit L we write $\lim_{n\to\infty} S_n = L$ (or) $S_n \to L$ as $n \to \infty$.

Examples:

(1). $\{1/n\} = \{1, 1/2, 1/3, \ldots\}$. The limit of the sequence $\{1/n\}$ is '0'.

(2) The limit of the sequence $\{1 - 1/n\}$ $= \{1 - 1/1, 1 - 1/2, 1 - 1/3, \ldots\}$ is '1'.

(3) The limit of the sequence $\{1 + 1/n\}$ $= \{1 + 1/1, 1 + 1/2, 1 + 1/3, \ldots\}$ is '1'.

Problems:

(1) S.T the sequence $1, 1/2, 1/3, \ldots$

(ie) $S_n = 1/n \ (n = 1, 2, 3, \ldots)$ has the limit '0'.

Soln: Let $\epsilon > 0$, we must find $N \in I$ so that $|S_n - L| < \epsilon$.

From this $|1/n - 0| < \epsilon, \ n \geq N \longrightarrow$ (1)

Thus if $|1/n| < \epsilon, \ n \geq N \longrightarrow$ (2)

Thus if we choose N such that $\frac{1}{N} < \epsilon$.

Since $\frac{1}{n} \leq \frac{1}{N} < \epsilon$ if $n \geq N$

Now $\frac{1}{N} < \epsilon$ if and only if $N > 1/\epsilon$.

Hence, if we take any $N \in I$ such that $N > 1/\epsilon$, $|S_n - 0| < \epsilon \ \forall \ n \geq N = 1/\epsilon$

$\therefore$ '0' is the limit of the sequence $\{1/n\}$

② S.T the sequence $\left\{\frac{2n}{n+4n^{1/2}}\right\}_{n=1}^{\infty}$ has the limit 2.

Soln:

Given $\epsilon > 0$, we must find $N \in I$ such that

$$\left|\frac{2n}{n+4n^{1/2}} - 2\right| < \epsilon, \ n \geq N \longrightarrow ①$$

$$\left|\frac{2n - 2n - 8n^{1/2}}{n+4n^{1/2}}\right| < \epsilon, \ n \geq N$$

$$\frac{8n^{1/2}}{n+4n^{1/2}} < \epsilon, \ n \geq N \longrightarrow ②$$

When n is large then n is much bigger than $n^{1/2}$. Then $4n^{1/2}$ can be ignored.

$\frac{8n^{1/2}}{n} = \frac{8}{n^{1/2}}$. Hence ② will be true, if

$$\frac{8}{n^{1/2}} < \epsilon, \ n \geq N \longrightarrow ③$$

If we choose N so that $\frac{8}{N^{1/2}} < \epsilon$,

(i.e.) choose $N > \frac{64}{\epsilon^2}$, then ③ will certainly be true.

$\therefore N > \frac{64}{\epsilon^2}$ is a positive integer.

$|S_n - 2| < \epsilon, \ \forall \ n \geq N = \frac{64}{\epsilon^2}$

$\therefore$ 2 is the limit of the given sequence.

③ By defn of Limit S.T $\lim_{n\to\infty} \frac{2n}{n+3} = 2$.

Soln:

For every $\epsilon > 0$, we must find $N \in I$ such that $|S_n - 2| < \epsilon, \ n \geq N$.

$$\left|\frac{2n}{n+3} - 2\right| < \epsilon, \ n \geq N$$

$$\left|\frac{2n-2n-6}{n+3}\right| < \epsilon,\ n \geq N$$

$$\frac{6}{n+3} < \epsilon,\ n \geq N$$

$$\frac{6}{\epsilon} < n+3,\ n \geq N$$

$$\frac{6}{\epsilon} - 3 < n,\ n \geq N$$

$\therefore N > \frac{6}{\epsilon} - 3$ is a positive integer such that $|S_n - 2| < \epsilon,\ n \geq N = \frac{6}{\epsilon} - 3$

$\therefore$ 2 is the limit of the sequence

(4) By defn of limit S.T $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$.

Proof:

For every $\epsilon > 0$, there exists $N \in I$ such that $|S_n - 0| < \epsilon,\ n \geq N$

$$\left|\frac{1}{\sqrt{n+1}} - 0\right| < \epsilon,\ n \geq N$$

$$\frac{1}{\sqrt{n+1}} < \epsilon,\ n \geq N$$

$$\frac{1}{n+1} < \epsilon^2,\ n \geq N$$

$$\frac{1}{\epsilon^2} < n+1,\ n \geq N.$$

$$\frac{1}{\epsilon^2} - 1 < n,\ n \geq N.$$

$N > \frac{1}{\epsilon^2} - 1$ is a positive integer such that $|S_n - 0| < \epsilon,\ n \geq N > \frac{1}{\epsilon^2} - 1$.

$\therefore$ 0 is the limit of the given sequence.

Theorem: 1.6

3. If $\{S_n\}_{n=1}^{\infty}$ is a sequence of non-negative numbers and if $\lim_{n\to\infty} S_n = L$ then $L \geq 0$.

Proof:

Let $L < 0$, Then take $\epsilon = -\frac{L}{2}$ there exists $N \in I$ such that

$$|S_n - L| < -\frac{L}{2},\ n \geq N$$

$$\frac{L}{2} < S_n - L < -\frac{L}{2},\ n \geq N$$

$$\frac{L}{2} + L < S_n < -\frac{L}{2} + L$$

$$\frac{3L}{2} < S_n < \frac{L}{2} < 0$$

$$\therefore S_n < \frac{L}{2} < 0$$

by hypothesis $S_n \geq 0$.

This implies $L > 0$, contradicting our assumption that $L < 0$. Hence $L \geq 0$

Note: (Hint)

About defn of limit – ϵ – defn ϵ measures the error between the numbers x_n and the limit L while the integer N measures how fast the sequence gets closer to the limit L. (or) tells how far you have to go to get closer to L upto ϵ.

Subsequence:

If $S = \{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $N = \{n_i\}_{i=1}^{\infty}$ is a subsequence of the sequence of positive integers, then the composite function $S \circ N$ is called the subsequence of S.

For $i \in I$ we have $N(i) = n_i$

$$S \circ N(i) = S[N(i)] = S(n_i) = S_{n_i}$$

and hence $S \circ N = \{S_{n_i}\}_{n=1}^{\infty}$

Examples:

1) The sequence of primes $2, 3, 5, 7, 11, \dots$
2) The set of even nos $2, 4, 6, 8, \dots$
3) The set of odd nos $1, 3, 5, \dots$

Hint: $N = 1, 2, 3, 4, 5, \dots$ without changing the order if any subset can be taken out from N then that is called subsequence.

Problems:

1 If $S = \{S_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty}$ and $N = \{n_i\}_{i=1}^{\infty} = \{i^2\}_{i=1}^{\infty}$ find S_5, S_9, n_2, S_{n_3}

Soln:

$S_5 = 2(5) - 1 = 9$

$S_9 = 2(9) - 1 = 17$

$n_2 = n(2) = 2^2 = 4$, $n_3 = n(3) = 3^2 = 9$

$S_{n_3} = S(n_3) = S(9) = 2(9) - 1 = 17$

Convergent Sequences:

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ has the limit L, we say that $\{S_n\}_{n=1}^{\infty}$ is convergent to L. If $\{S_n\}_{n=1}^{\infty}$ does not have a limit, we say that $\{S_n\}_{n=1}^{\infty}$ is divergent.

Example:

1) The sequence $\{1, 1/2, 1/3, \dots\} = \{1/n\}$ converges to '0'.

   $\frac{(-1)^n}{n}$, $1+\frac{1}{n^2}$ are convergent.

2) The sequences $\{1, 2, 3, \dots\} = \{n\}$ and $\{-1, +1, -1, +1, \dots\} = (-1)^n$ are divergent sequences.

   $1+(-1)^n$, n^2 are divergent.

Note: The sequence cannot converge to more than one limit.

Theorem: 1.7

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent to L, then $\{S_n\}_{n=1}^{\infty}$ cannot converge to a limit distinct from L.

(i) if $\lim_{n\to\infty} S_n = L$ and $\lim_{n\to\infty} S_n = M$ then $L = M$.

Proof:

Assume the contrary $L \neq M$.
So that $|M-L| > 0$. Let $\epsilon = \frac{|M-L|}{2}$.
By hypothesis $\lim_{n\to\infty} S_n = L$ there exists $N_1 \in I$ such that

$$|S_n - L| < \frac{1}{2}|M-L|, \quad n \geq N_1$$

Similarly, since $\lim_{n\to\infty} S_n = M$ there exists $N_2 \in I$ such that

$$|S_n - M| < \frac{|M-L|}{2}, \quad n \geq N_2$$

Let $N = \max\{N_1, N_2\}$

$$|M-L| = |M-S_n+S_n-L|$$
$$\leq |M-S_n|+|S_n-L|$$
$$< \frac{|M-L|}{2} + \frac{|M-L|}{2}$$

$|M-L| < |M-L|$ is a contradiction.

$\therefore L=M$. Hence the theorem.

Theorem: 1.5

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent to L then any subsequence of $\{S_n\}_{n=1}^{\infty}$ is also convergent to L.

Proof: Let $\{S_{n_k}\}$ be a subsequence of $\{S_n\}$

Since $\lim_{n\to\infty} S_n = L$ we get for $\epsilon>0$,

$\exists N>0$ such that $|S_n-L|<\epsilon, \forall n \geq N$.

Let n_k be the subsequence of the sequence of +ve integers.

$\therefore$ for every $\epsilon>0 \ \exists N$ s.t $|S_{n_k}-L|<\epsilon$ $\forall n_k \geq N$.

Divergent Sequences:

Sequence diverges to ∞:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that S_n approaches ∞ as n approaches infinity if for any real number $M>0$ there is a positive integer N such that $S_n \geq M$, $n \geq N$.

In this case we write $\lim_{n\to\infty} S_n = \infty$ and we say that $\{S_n\}_{n=1}^{\infty}$ diverges to infinity.

Sequences diverges to $-\infty$:

Let $\{S_n\}$ be a sequence of real numbers. We say that S_n approaches $-\infty$ as n approaches infinity if, for any real $M>0$, there is a +ve integer N, such that $S_n<-M$, $n\geq N$

In this case we write $\lim_{n\to\infty} S_n = -\infty$.
and we say that $\{S_n\}_{n=1}^{\infty}$ diverges to $-\infty$.

Example:

1. The sequence $\{-1, -2, \dots\}$ diverges to $-\infty$.
2. The sequence $\{1, 2, 3, \dots\}$ diverges to ∞.
3. The sequence $1, -2, 3, -4, \dots$ does not approach either to infinity or minus infinity. However, this sequence has a subsequence $1, 3, 5, \dots$ which approaches to infinity and also has a subsequence $-2, -4, -6, \dots$ which approaches to $-\infty$.

Oscillatory Sequence:

If the sequence $\{S_n\}_{n=1}^{\infty}$ of real numbers diverges but does not diverge to either infinity or to minus infinity we say that $\{S_n\}_{n=1}^{\infty}$ oscillates.

Example:

$\{(-1)^n\}$ is oscillatory.

Problems:

1. P.T the sequence $\{\log 1/n\}$ diverges to $-\infty$.

Soln:

For any real $M > 0$, there is a +ve integer N such that

$$\log(1/n) < -M, \quad n \geq N$$
$$\log 1 - \log n < -M$$
$$-\log n < -M$$
$$\log n > M$$
$$n > e^M.$$

Thus we choose $N > e^M$ for which the sequence $\{\log 1/n\}$ diverges to $-\infty$.

---

2. Verify whether the sequence $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$ converges, diverges or oscillates.

Soln:

$$\left\{\sin\left(\frac{n\pi}{2}\right)\right\} = \{1, 0, -1, 0, \dots\}$$

$\therefore$ The sequence $\left\{\sin\frac{n\pi}{2}\right\}$ is an oscillatory sequence.

3. Verify whether the sequence $\left\{n\sin\left(\frac{n\pi}{2}\right)\right\}$ converges, diverges or oscillates.

Soln:

$$\left\{n\sin\left(\frac{n\pi}{2}\right)\right\} = \{1, 2(0), 3(-1), 0, \dots\}$$

$$= \{1, 0, -3, 0, -5, \dots\}$$

$\therefore$ The sequence $\left\{n\sin\frac{n\pi}{2}\right\}$ is a divergent sequence.

4. P.T $\{n\}$ is divergent to ∞.

Soln: Given any real number $M>0$ we can find an integer N s.t, $n \geq M$ $\forall\ n \geq N$.

Thus we choose $N \geq M$ for which the sequence $\{n\}$ diverges to ∞.

Note: If the sequence of real number $\{S_n\}_{n=1}^{\infty}$ converges to L then the sequence $\{|S_n|\}_{n=1}^{\infty}$ converges to $|L|$. prove that the converse is not true by giving an example.

Soln:

$\{S_n\}_{n=1}^{\infty}$ converges to L. For every $\epsilon>0$ there exists a +ve number N such that $|S_n - L| < \epsilon$, $n \geq N$.

W.K.T $\quad ||a|-|b|| \leq |a-b|$

$$||S_n|-|L|| \leq |S_n - L| < \epsilon, \ n \geq N$$

$\therefore \lim_{n\to\infty} |S_n| = |L| \quad \therefore \{|S_n|\}$ cgs to $|L|$

Its converse is not true.

If $\{|S_n|\}$ converges to $|L|$ then $\{S_n\}$ doesnot converge to the limit L.

Ex: $\{S_n\} = \{1, -1, 1, -1, \dots\}$

$\{|S_n|\} = \{1, 1, 1, \dots\}$

$\{|S_n|\}$ converges to 1. But $\{S_n\}$ is not cgt.

Cartesian product:

Let $A = \{x_1, x_2, \dots\}$ and $B = \{y_1, y_2, y_3, \dots\}$

Then the cartesian product is given by

$$A \times B = \begin{cases} (x_1, y_1), (x_1, y_2), \dots \\ (x_2, y_1), (x_2, y_2), \dots \\ \vdots \end{cases}$$

① If A and B are countable S.T $A \times B$ is also countable.

② If Z is the set of integers S.T $Z \times Z$ is countable.

③ The set $N \times N$ is countable.

Soln: We may arrange the set $N \times N$ as shown below.

$$\begin{array}{ccccc} (1,1) \rightarrow & (1,2) & (1,3) \rightarrow & (1,4) & \dots \\ (2,1) & (2,2) & (2,3) & (2,4) & \dots \\ (3,1) & (3,2) & (3,3) & (3,4) & \dots \\ (4,1) & (4,2) & (4,3) & (4,4) & \dots \\ \vdots & & & & \end{array}$$

We may arrange the elements in the order indicated by the arrows. This scheme arranges all the elements of $N \times N$ into a sequence and hence $N \times N$ is countable.

4. $[0,1]$ is uncountable. Then S.T $[1,2]$ is also uncountable.

Soln: Let $f : [0,1] \rightarrow [1,2]$ Consider $f(x) = x+1$

The fun f is 1-1 and onto. But $[0,1]$ is uncountable. $\therefore [1,2]$ is also uncountable.