19PMA11

CALCULUS OF VARIATIONS AND INTEGRAL EQUATIONS

M.Sc MATHEMATICS

III - SEMESTER

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UNIT - II variational problems with moving Boundaries.

Movable boundary for a functional dependent on to functionals - one side variations - reflection and refraction of extremes - difraction of light rays [chapter - 2 ; sec 2.1 - 2.5] .

UNIT – I

problems:-

Find the curve which is an extremum of the function $I=\int_0^{\pi/4}\left[y^2-\left(\frac{dy}{dx}\right)^2\right]dx$, $y(0)=0$, we attained if the second boundary point is permitted to move along the straight line, $x=\pi/4$.

Soln:- Given that,

$$I=\int_0^{\pi/4}\left[y^2-\left(\frac{dy}{dx}\right)^2\right]dx$$

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \rightarrow (1)$$

Here $F=y^2-y'^2$

$$\frac{\partial F}{\partial y}=2y,\quad \frac{\partial F}{\partial y'}=-2y'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=-2y''$$

$$(1)\Rightarrow 2y+2y''=0$$

$$y''+y=0.$$

This can be written as,

$$(D^2+1)y=0$$

The A.E is, $m^2+1=0$

$$m^2=-1$$

$$m=\pm i$$

The C.F is $y=A\cos x+B\sin x$

$$y(0)=0\Rightarrow A\cos 0+B\sin 0=0$$

$$A=0.$$

$$\therefore y=B\sin x$$

By natural boundary condition.

$$\left[\frac{\partial F}{\partial y'}\right]_{x=\pi/4}=0.$$

$$\Rightarrow\left[-2y'\right]_{x=\pi/4}=0.$$

$$-2(B\cos x)_{x=\pi/4}=0.$$

$$-2B\cos \pi/4 = 0$$

$$-2B\left[\frac{1}{\sqrt{2}}\right]=0$$

$$-\sqrt{2}B=0$$

$$B=0$$

Sub in ③ $\Rightarrow y=0$.

Hence the proof.

2). Find the ~~transverlity~~ transversality condition for the function

$$V=\int_{x_0}^{x_1} A(x,y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx.$$

Soln:- Given that

$$V=\int_{x_0}^{x_1} A(x,y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx.$$

Transverlity condition.

$$\boxed{F+(\phi'-y')\frac{\partial F}{\partial y'}=0} \rightarrow ① \quad 2m \quad 3m$$

Here $F=A(x,y)\sqrt{1+y'^2}$

$$\frac{\partial F}{\partial y'}=A\times\frac{1}{2}(1+y'^2)^{-1/2}\times 2y'$$

$$=Ay'(1+y'^2)^{\frac{-1}{2}}$$

$$\frac{\partial F}{\partial y'}=\frac{Ay'}{\sqrt{1+y'^2}}$$

$$① \Rightarrow A\sqrt{1+y'^2}+(\phi'-y')\frac{Ay'}{\sqrt{1+y'^2}}=0.$$

$$A(1+y'^2)+(\phi'-y')Ay'=0$$

$$A+Ay'^2+\phi'Ay'-Ay'^2=0$$

$$A+\phi'Ay'=0.$$

$$A = -Ay'\phi'$$

$$y' = \frac{A}{-A\phi'}$$

$$y = \frac{-1}{\phi'}$$

This is orthogonality condition.//

3). Find the shortest distance b/w the parabola $y = x^2$ and straight line $x - y = 5$.

Soln:-

The problem is to find the extremum value of the given functional.

$$I[y(x)] = \int_{x_0}^{x_1} \sqrt{1+y'^2}\,dx = 0$$

Subject to the condition left end moves along the curve $y = x^2$ and right end moves along the straight line $x - y = 5$,

the transversality condition is,

$$F + (\phi' - y')\frac{\partial F}{\partial y'} = 0 \rightarrow ①$$

Here $F = \sqrt{1+y'^2}$

$$\frac{\partial F}{\partial y'} = \frac{1}{\cancel{2}}(1+y'^2)^{-1/2}\cancel{2}y'$$

$$= \frac{y'}{\sqrt{1+y'^2}}$$

Then $y = x^2$

Let $\phi = x^2 \Rightarrow \phi' = 2x$

eqn ①, becomes,

$$\sqrt{1+y'^2} + (2x - y')\frac{y'}{\sqrt{1+y'^2}} = 0 \rightarrow ②$$

$$(1+y'^2) + (2x - y')y' = 0$$

$$1+y'^2+2xy'-y'^2=0$$
$$1+2xy'=0$$
$$2xy'=-1$$
$$\boxed{y'=\frac{-1}{2x}}$$

And $x-y=5$
$$y=x-5$$
$$\phi=x-5$$
$$\phi'=1$$

eqn (1), becomes.

$$\sqrt{1+y'^2}+(1-y')\frac{y'}{\sqrt{1+y'^2}}=0 \longrightarrow (3)$$
$$(1+y'^2)+(1-y')y'=0$$
$$1+\cancel{y'^2}+y'-\cancel{y'^2}=0$$
$$1+y'=0$$
$$\boxed{y'=-1=A}$$

By Euler's eqn

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]=0 \longrightarrow (4)$$

$$\frac{\partial F}{\partial y}=0, \quad \frac{\partial F}{\partial y'}=\frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]=\frac{\sqrt{1+y'^2}\cdot y''-y'\times\frac{1}{2}(1+y'^2)^{-1/2}\,y'\cdot y''}{(1+y'^2)}$$

$$=\frac{\sqrt{1+y'^2}\cdot y''-y'^2\cdot\frac{y''}{\sqrt{1+y'^2}}}{(1+y'^2)}=\frac{(1+y'^2)y''-y'^2y''}{(1+y'^2)^{3/2}}$$

$$=\frac{y''+y'^2y''-y'^2y''}{(1+y'^2)^{3/2}}=\frac{y''}{(1+y'^2)^{3/2}}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=\frac{y''}{(1+y'^2)^{3/2}}$$

eqn (4), becomes,

$$\textcircled{1} \Rightarrow \frac{y''}{(1+y'^2)^{3/2}} = 0$$

$$y'' = 0$$

Integrating $y' = A$.

$$y = Ax + B \longrightarrow \textcircled{5}$$

$$(x_1, y_1) \Rightarrow y_1 = Ax_1 + B$$

$$y_1 = x_1^2 \Rightarrow x_1^2 = Ax_1 + B. \rightarrow *$$

$$(x_2, y_2) \Rightarrow y_2 = Ax_2 + B$$

$$y_2 = x_2 - 5 \Rightarrow x_2 - 5 = Ax_2 + B.$$

eqn ②, implies $\rightarrow y' = A$

$$\sqrt{1+A^2} + (2x_1 - y')\frac{A}{\sqrt{1+A^2}} = 0$$

$$(1+A^2) + (2x_1 - A)A = 0$$

$$1 + \cancel{A^2} + 2x_1 A - \cancel{A^2} = 0$$

$$1 + 2x_1 A = 0$$

$$2x_1 A = -1$$

$$\boxed{A = \frac{-1}{2x_1}} \rightarrow \textcircled{6}$$

eqn ③ becomes,

$$\sqrt{1+A^2} + (1-A)\frac{A}{\sqrt{1+A^2}} = 0$$

$$(1+A^2) + (1-A)A = 0$$

$$1 + A^2 + A - A^2 = 0$$

$$1 + A = 0$$

$$y' = \boxed{A = -1} \rightarrow \textcircled{7}$$

equating ⑥ and ⑦

$$-1 = \frac{-1}{2x_1}$$

$$x_1 = \frac{1}{2} \rightarrow \text{eqn } *$$

$$\Rightarrow x_1^2 = Ax_1 + B$$

$$B = x_1^2 - Ax_1.$$

Sub $x_1 = \frac{1}{2}$, we get.

$$B = \left(\frac{1}{2}\right)^2 - (-1)\left(\frac{1}{2}\right)$$
$$= \frac{1}{4} + \frac{1}{2}$$
$$B = \frac{3}{4}$$

Then now,

$$x_2 - 5 = Ax_2 + B$$
$$x_2 - 5 = (-1)x_2 + \frac{3}{4}$$
$$x_2 - 5 = -x_2 + \frac{3}{4}$$
$$x_2 + x_2 = 5 + \frac{3}{4}$$
$$2x_2 = \frac{23}{4}$$
$$x_2 = \frac{23}{8}$$

$$S = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$
$$= \int_{1/2}^{23/8} \sqrt{1+(-1)^2}\, dx \quad = \int_{1/2}^{23/8} \sqrt{2}\, dx.$$
$$= \sqrt{2}\,[x]_{1/2}^{23/8} \quad = \sqrt{2}\left[\frac{23}{8} - \frac{1}{2}\right] = \sqrt{2}\left(\frac{19}{8}\right).$$

The shortest distance.

$$\boxed{S = \frac{19\sqrt{2}}{8}}$$

Hence completed.

4). $y^2 = x$. point $(-1, 5)$ find Shortest distance.

Soln:-

Let S be the length of the arc of the curve $y = f(x)$ b/w the curve (x_0, y_0) and (x_1, y_1).

$$S = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, dx.$$

Here the left end (x_0, y_0) is fixed at the point $(-1, 5)$. Here the right end (x_1, y_1) move along the curve $y^2 = x$.

$$I = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, dx.$$

The Euler eqn is,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0 \rightarrow ①$$

$$F = \sqrt{1+y'^2}$$

$$\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = \frac{\sqrt{1+y'^2}\, y'' - y' \times \frac{1}{2}(1+y'^2)^{-1/2} \cdot 2y' \cdot y''}{(1+y'^2)}$$

$$= \frac{y''\sqrt{1+y'^2} - \frac{y''y'^2}{\sqrt{1+y'^2}}}{(1+y'^2)} = \frac{y''\sqrt{1+y'^2} - \frac{y''y'^2}{\sqrt{1+y'^2}}}{(1+y'^2)}$$

$$= \frac{y''(1+y'^2) - y''y'^2}{(1+y'^2)^{3/2}}$$

$$= \frac{y'' + \cancel{y''y'^2} - \cancel{y''y'^2}}{(1+y'^2)^{3/2}} \Rightarrow \frac{y''}{(1+y'^2)^{3/2}}$$

Now eqn ①, becomes,

$$0 - \frac{y''}{(1+y'^2)^{3/2}} = 0$$

$$y'' = 0$$

$$y' = A$$

$$y = Ax + B$$

$$(x_0, y_0) \Rightarrow y_0 = Ax_0 + B.$$

$$(-1, 5) \Rightarrow \boxed{5 = -A + B}$$

$$(x_1, y_1) \Rightarrow y_1 = Ax_1 + B$$

$y = \sqrt{x}$

$\phi = \sqrt{x}$

$\phi' = \frac{1}{2\sqrt{x}}.$

Using transversality condition,

$y = \sqrt{x}$

$\phi' = y' = \frac{1}{2\sqrt{x}}$

$$F + (\phi' - y')\frac{\partial F}{\partial y'} = 0.$$

$$\sqrt{1+y'^2} + \left(\frac{1}{2\sqrt{x}} - y'\right)\frac{y'}{\sqrt{1+y'^2}} = 0$$

$$(1+y'^2)+\left(\frac{1}{2\sqrt{x}}-y'\right)y'=0$$

$$1+y'^2+\frac{y'}{2\sqrt{x}}-y'^2=0$$

$$1+\frac{y'}{2\sqrt{x}}=0$$

$$\frac{y'}{2\sqrt{x}}=-1$$

$$y'=-2\sqrt{x}$$

$y_1'=A$ at $x=x_1,\ y=y_1$

$$y_1'=-2\sqrt{x}$$

$$A=-2\sqrt{x_1}$$

using ②,

$$5=-A+B$$

$$5=2\sqrt{x_1}+B$$

$$B=5-2\sqrt{x_1}$$

$-2\sqrt{x_1}+\sqrt{x_1}$

$-\sqrt{x_1}$

To find x_1 :-

$$y_1=Ax_1+B$$

$\sqrt{x_1}\cdot x_1$

$x_1^{3/2}$

$$\sqrt{x_1}=(-2\sqrt{x_1})x_1+(5-2\sqrt{x_1})$$

$$\sqrt{x_1}=-2(x_1)^{3/2}+5-2\sqrt{x_1}$$

$$-2(x_1)^{3/2}+5-2\sqrt{x_1}-\sqrt{x_1}=0$$

$$-2x_1^{3/2}+5-3x_1^{1/2}=0$$

$-2x_1^{3/2}-3x_1^{1/2}$

$x_1^{1/2}(-2x_1-3)$

$$-5x_1^{3/2+1/2}+5=0$$

$$\Rightarrow x_1^2=1$$

$$x_1=1$$

$$y_1=\sqrt{x_1}=1$$

$$S=\int_{x_0}^{x_1}\sqrt{1+y'^2}\,.dx$$

$$=\int_{-1}^{1}\sqrt{1+(-2)^2}\,.dx$$

$$=\int_{-1}^{1}\sqrt{5}\,.dx$$

$= \sqrt{5}[x]_{-1}^{1} \Rightarrow \sqrt{5}[1-(-1)]$

$S = 2\sqrt{5}$

Hence the Shortest distance $S = 2\sqrt{5}$.

(12) a). 5m

5). Find the extremals of the functional

$$I = \int_{x_1}^{x_2} (y'^2 + z'^2 + 2yz)\,dx.$$

$y(0)=0,\ z(0)=0$ at the point (x_2, y_2, z_2) moves along fixed plane $x = x_2$.

Soln:-

Given that,

$$I = \int_{x_1}^{x_2} (y'^2 + z'^2 + 2yz)\,dz.$$

Euler eqn is,

$$\left.\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0 \\ \frac{\partial F}{\partial z} - \frac{d}{dx}\left[\frac{\partial F}{\partial z'}\right] = 0 \end{aligned}\right\} \rightarrow (1)$$

$$F = y'^2 + z'^2 + 2yz$$

$$\frac{\partial F}{\partial y} = 2z, \quad \frac{\partial F}{\partial y'} = 2y'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$$

$$\frac{\partial F}{\partial z} = 2y, \quad \frac{\partial F}{\partial z'} = 2z'$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 2z''$$

Eqn (1), becomes,

$$2z - 2y'' = 0, \quad 2y - 2z'' = 0$$

$$y'' - z = 0, \quad z'' - y = 0$$

$$D^2 y - z = 0 \rightarrow (2). \quad D^2 z - y = 0 \rightarrow (3).$$

$$(2) \times D^2 \Rightarrow D^4 y - \cancel{D^2 z} = 0$$

$$(3) \Rightarrow \quad y + \cancel{D^2 z} = 0$$

$$D^4 y - y = 0$$

The A.E is

$$m^4 - 1 = 0$$

$$m^4 = 1$$

$$m = \pm 1, \pm i$$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x \rightarrow (4)$$

$$y' = c_1 e^{x} - c_2 e^{-x} - c_3 \sin x + c_4 \cos x \rightarrow (5)$$

$$z = y'' = c_1 e^{x} + c_2 e^{-x} - c_3 \cos x - c_4 \sin x \rightarrow (6)$$

$$z' = c_1 e^{x} - c_2 e^{-x} + c_3 \sin x - c_4 \cos x \rightarrow (7)$$

$$y(0) = 0 \Rightarrow 0 = c_1 + c_2 + c_3 \rightarrow (A)$$

$$z(0) = 0 \Rightarrow 0 = c_1 + c_2 - c_3$$

Adding the above two.

$$2c_1 + 2c_2 = 0$$

$$\Rightarrow c_1 + c_2 = 0$$

$$c_1 = -c_2$$

Sub in (A)

$$\Rightarrow c_3 = 0$$

For the condition at the moving boundary point (x_2, y_2, z_2) can be derived with $\delta x_2 = 0$.

$$(F_{y'})_{x=x_2} = 0, \quad (F_{z'})_{x=x_2} = 0$$

$$\left(\frac{\partial F}{\partial y'}\right)_{x=x_2} = 0 \qquad \left(\frac{\partial F}{\partial z'}\right)_{x=x_2} = 0$$

$$(2y')_{x=x_2} = 0 \qquad (2z')_{x=x_2} = 0$$

$$(y') = 0 \qquad (z')_{x=x_2} = 0$$

$$y'(x_2) = 0. \qquad z'(x_2) = 0$$

(5) $\Rightarrow y'(x_2) = c_1 e^{x_2} - c_2 e^{-x_2} - c_3 \sin x_2 + c_4 \cos x_2 = 0$

(7) $\Rightarrow z'(x_2) = c_1 e^{x_2} - c_2 e^{x_2} + c_3 \sin x_2 - c_4 \cos x_2 = 0.$

$\boxed{c_1 = -c_2}$ and $c_3 = 0.$

$$-c_2 e^{x_2} - c_2 e^{-x_2} + c_4 \cos x_2 = 0 \rightarrow (8)$$
$$-c_2 e^{x_2} - c_2 e^{-x_2} - c_4 \cos x_2 = 0 \rightarrow (9)$$

eqn (8), becomes.

$$-c_2 (e^{x_2} + e^{-x_2}) + c_4 \cos x_2 = 0.$$
$$c_2 (e^{x_2} + e^{-x_2}) - c_4 \cos x_2 = 0 \rightarrow (10)$$

eqn (9), becomes,

$$-c_2 (e^{x_2} + e^{-x_2}) + c_4 \cos x_2 = 0.$$
$$c_2 (e^{x_2} + e^{-x_2}) + c_4 \cos x_2 = 0 \rightarrow (11)$$

(10) + (11)

$$2c_2 (e^{x_2} + e^{-x_2}) = 0$$
$$4c_2 \cosh x_2 = 0 \qquad \left[\because \frac{e^{\theta} + e^{-\theta}}{2} = \cosh\theta\right.$$
$$\boxed{c_2 = 0}$$

(A) $\Rightarrow$ $c_1 = 0$

$$y = c_4 \sin x$$
$$z = -c_4 \sin x$$

(12) b) 8m Hence prove.

(6). Find the shortest path from the point $A(-2,3)$ to the point $B(2,3)$ located in the region $y \leq x^2$

Soln:-

The problem is to find the extremal of the functional

$$I(y) = \int_{-2}^{2} \sqrt{1+y'^2}\, dx \rightarrow (1)$$

Subject to the condition

$$y(-2) = 3 \text{ and } y(2) = 3$$

The Euler eqn is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \rightarrow (2)$$

$$F = \sqrt{1+y'^2}$$

$$\frac{\partial F}{\partial y}=0, \quad \frac{\partial F}{\partial y'}=\frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=\frac{y''}{(1+y'^2)^{3/2}}$$

② becomes.

$$\frac{-y''}{(1+y'^2)^{3/2}}=0$$

$\Rightarrow y''=0 \Rightarrow y'=A$

$y=Ax+B \rightarrow$ Ⓐ

$y(-2)=3 \Rightarrow 3=-2A+B$

$B=3+2A \rightarrow$ ③

$y(2)=3 \Rightarrow 3=2A+B$

$B=3-2A \rightarrow$ ④

③ + ④ $\Rightarrow 2B=6$

$\Rightarrow B=3$

Sub $B=3$ in ④

$A=0$

Sub $A=0, B=3$ in eqn Ⓐ

$y=3$

If ye integrant is I_y then $F_{y'y'}$, $\frac{\partial F}{\partial y'}=\frac{y'}{\sqrt{1+y'^2}}$

$$\frac{d}{dy'}\left(\frac{\partial F}{\partial y'}\right)=\frac{(1+y'^2)^{1/2}-\frac{1}{2}(1+y'^2)^{-1/2}\cdot y' 2y'}{(1+y'^2)}$$

$$=\frac{(1+y'^2)^{1/2}-y'^2(1+y'^2)^{-1/2}}{(1+y'^2)}$$

$$=\frac{(1+y'^2)-y'^2}{(1+y'^2)^{3/2}}=\frac{1}{(1+y'^2)^{3/2}}$$

$$=(1+y'^2)^{-3/2}$$

which is not equal to zero.

The lines every P & Q tangent to the parabola $y=x^2$ and the position PQ of the parabola.

Let the Ab of P & Q be $-\bar{x}$ and $+\bar{x}$ respectively,

Let the condition of tangency at Q.

$$y = Ax+B$$
$$x^2 = Ax+B$$
$$\bar{x}^2 = A\bar{x}+B \rightarrow 5$$

At a point P.

$$\bar{x}^2 = -A\bar{x}+B$$
$$\bar{x}^2 = B-A\bar{x} \rightarrow 6$$

5 and 6

$$\bar{x}^2 = \cancel{A\bar{x}}+B$$
$$\bar{x}^2 = \cancel{-A\bar{x}}+B$$
$$2\bar{x}^2 = 2B$$
$$B = \bar{x}^2$$

eqn 5 becomes,

$$\bar{x}^2 = Ax + \bar{x}^2$$
$$A=0$$

Since the tangent PQ passes through

$$y=Ax+B$$
$$3 = 2A+B$$

put $\boxed{A=0}$

$$B=3, B=\bar{x}^2=3$$
$$\bar{x}=\sqrt{3}$$

$$y = Ax+B \Rightarrow 0+3$$
$$\boxed{y=3}$$

7). Find the extremal with corner points of the functions

$$I[y(x)] = \int_{x_1}^{x_2} y'^2 (1-y')^2 \, dx$$

Soln:

Given that $I[y(x)] = \int_{x_1}^{x_2} y'^2 (1-y'^2) \, dx$

Here $F = y'^2 (1-y')^2$

The Euler eqn is,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\frac{\partial F}{\partial y} = 0$$

$$\frac{\partial F}{\partial y'} = y'^2 \cdot 2(1-y')(-1) + (1-y')^2 \cdot 2y'.$$

$$= -2y'^2(1-y') + 2y'(1-y')^2$$

$$= 2y'(1-y')[-y' + (1-y')]$$

$$= 2y'(1-y')(1-2y')$$

$\frac{\partial F}{\partial y'} = F_{y'}$

$$\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 2y'(1-y')(-2y'') + 2(1-2y')y'(-y'') + 2(1-y')(1-2y')y''$$

$$= -4y'y'' + 4y'^2y'' - 2y'y'' + 4y'^2y'' + 2y'' - 2y'y'' - 4y'y'' + 4y'^2y''$$

$$= -12y'y'' + 12y'^2y'' + 2y''$$

Euler eqn becomes,

$$-(-12y'y'' + 12y'^2y'' - 2y'') = 0$$

$$12y'y'' - 12y'^2y'' + 2y'' = 0$$

$$2y''[-6y' + 6y'^2 + 1] = 0$$

$$y''[-6y' + 6y'^2 + 1] = 0$$

$$y'' = 0$$

$$\Rightarrow y' = A$$

$$\Rightarrow y = Ax + B$$

$\int \partial y \quad y' = A$

$y = Ax + B.$

Now at a corner point the condition becomes,

$$(F - y'F_{y'})_{x=\bar{x}-0} = (F - y'F_{y'})_{x=\bar{x}+0}$$

$$[y'^2(1-y')^2 - y'2y'(1-y')(1-2y')]_{x=\bar{x}-0} = [y'^2(1-y')^2 - y'2y'(1-y')(1-2y')]_{x=\bar{x}+0}$$

$$[y'^2(1-y')(1-y'-2(1-2y'))]_{x=\bar{x}-0} = [y'^2(1-y')(1-y'-2(1-2y'))]_{x=\bar{x}+0}$$

$$[y'^2(1-y')(1-y'-2+4y')]_{x=\bar{x}-0} = [y'^2(1-y')(1-y'-2+4y')]_{x=\bar{x}+0}$$

$$[y'^2(1-y')(3y'-1)]_{x=\bar{x}-0} = [y'^2(1-y')(3y'-1)]_{x=\bar{x}+0}$$

$$(y')_{x=\bar{x}-0} = (y')_{x=\bar{x}+0}$$

$$y'(\bar{x}-0) = y'(\bar{x}+0)$$

$$y'(\bar{x}-0) = 0, \quad y'(\bar{x}+0) = 1.$$

Thus the broken line extremal consist of straight line belong to

$$y = A \text{ and } y = x+B$$

$$F_{y'} = 2y'(1-y')(-1-2y')$$

which may vanishes,

This is possible induction of existence of the corner point.

Hence completed.

8). Using only the basic necessary condition $\delta I = 0$ find the curve on which an extremals of the functional.

$$I[y(x)] = \int_0^{x_1} \frac{(1+y_1^2)^{1/2}}{y}\, dx, \quad y(0) = 0 \text{ can be}$$

acheived in the 2nd boundary point x_1, y_1 can more along the circumference $(x-9)^2 + y^2 = 9$.

Soln:-

Given that,

$$F = \frac{1}{y}(1+y'^2)^{1/2}$$

Euler eqn is $\qquad F - y'F_{y'} = c$

$$F - y'F_{y'} = c$$

$$\frac{1}{y}(1+y'^2)^{1/2} - y'\left(\frac{y'}{y\sqrt{1+y'^2}}\right) = c$$

Taking L.C.M

$$1+\cancel{y'^2}-\cancel{y'^2} = c \cdot y\sqrt{1+y'^2}$$

$$1 = cy\sqrt{1+y'^2}$$

$$1+y'^2 = \frac{1}{c^2y^2}$$

$$1+y'^2 = \frac{c_1}{y^2}$$

Consider $x^2+y^2 =$ some constant, $\qquad (x-c_1)^2+(y-0)^2=c_2^2$

$(x-c_1)^2+y^2=c_2^2$

$$(x-c_1)^2+(y-0)^2 = c_2^2$$

$$(x-c_1)^2+y^2 = c_2^2 \longrightarrow (1)$$

The boundary condition.

$$y(0)=0 \Rightarrow x=0=y$$

$$(1) \Rightarrow c_1^2 = c_2^2$$

$$c_1 = c_2$$

The boundary condition.

$$y(0)=0$$

$$\text{i.e.,}\ c_1 = c_2$$

Further since the integral of the form $A(x,y)(1+y'^2)^{1/2}$. the transversality condition at the movable boundary point x_1, x_2 reduces to orthogonality condition.

Thus the required extremals will be the arc of circle belonging to (1), which is orthogonal to $(x-9)^2+y^2=9$

Since $B(x_1,y_1)$ lies on both circle we must have,

$$(x-9)^2+y^2=9.$$
$$x^2+81-18x+y^2=9$$
$$x_1^2-18x_1+y_1^2=-72 \rightarrow (2)$$
$$(x-c_1)^2+y^2=c_2^2$$
$$x_1^2-2x_1c_1+y_1^2=0 \rightarrow (3)$$
(2) − (3) $\Rightarrow$
$$x_1^2-18x_1+y_1^2=-72$$
$$x_1^2-2x_1c_1+y_1^2=0$$
$$-18x_1+2x_1c_2=0-72$$
$$2x_1(-9+c_1)=-72$$
$$x_1(c_1-9)=-36 \rightarrow (4)$$
In view of orthogonality of two circles at (x_1, y_1)
The tangent to (1) at B passes through the Centre (9,0) of the given circle this yields.
$$(9-c_1)x_1=9c_1 \rightarrow (5)$$
(4) $\Rightarrow$ $x_1c_1-9x_1=-36 \rightarrow (6)$

(5) $\Rightarrow$ $9x_1-c_1x_1=9c_1$
$$9x_1-c_1x_1-9c_1=0 \rightarrow (7)$$
(6) + (7) $\Rightarrow$ $-9c_1=-36$
$$\boxed{c_1=4}$$
Put $c_1=4$ in equation (6).
$$4x_1-9x_1=-36$$
$$-5x_1=-36$$
$$\Rightarrow x_1=\frac{36}{5}$$
the value c_1 and x_1 in eqn (1)
$$(x-c_1)^2+y^2=c_2^2$$
$$(x-c_1)^2+y^2=c_1^2$$
$$(x-4)^2+y^2=c_1^2$$
$$x^2+16-8x+y^2=4^2$$

$$x^2+16-8x+y^2=16$$

$$y^2=8x-x^2$$

$$y=\pm\,(8x-x^2)^{1/2}.$$