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CALCULUS OF VARIATIONS AND INTEGRAL EQUATIONS

M.Sc MATHEMATICS

III - SEMESTER

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CALCULUS OF VARIATIONS and INTEGRAL EQUATIONS

UNIT-I variational problems with fixed boundaries

The concept of variation and its properties - Euler's equations - variational problems for functionals - functionals dependent on higher order derivatives - functions of several independent variables - some applications to problems of mechanics [chapter 1 - sec 1.1 - 1.7].

UNIT-II variational problems with moving Boundaries.

Movable boundary for a functional dependent on to functionals - one side variations - reflection and refraction of extremes - difraction of light rays [chapter -2 ; sec 2.1 - 2.5].

UNIT-III Integral equations

Introduction - Types of kernals - Eigen values and Eigen functions - connection with differential equation - solution of an integral equations - Initial value problems - boundary value problems. [chapter 1 . sec 1.1 - 1.3 & 1.5 - 1.8].

UNIT-IV. Solution of fredholm Integral equations

second kind with separable kernal. orthogonality and reality identi function - fredholm integral equation with separable kernal - solution of fredholm integral equation by successive substitution - successive aproximation - volterra integral equation -

- Solution by successive substitution.
[chapter 2, sec 2.1-2.3 and chapter 4, sec 4.1-

UNIT-V. Hilbert-Schmidt theory

complex Hilbert space - orthogonall syst of functions - gram-smidth orthogonallization process - Hilbert smidth theorems - solutions of fredholm integral equation of first kind.

[chapter 3. sec 3.1-3.4 and 3.8-3.9]

Text Book:

1. A.S. gupta - calculus of variations with applications - 2005 unit-1,
2. Sudir k. pundir and rimple pundir - Integral equation and boundary value problem

Unit - 1

Introduction:-

The calculus of Variation and integral deals with the problem of finding a function $y(x)$. such that a definite integral taken over a function $y(x)$. shall be maximum or minimum in otherwords calculus of Variation deals with maximaizing or minimizing the function.

Let $y_1(x)$ be the small variation of the curve $y(x)$. The closeness of $y(x)$ and the curve $y_1(x)$ is the absolute value of their difference. When (ie) $\{y(x), y_1(x)$ is small for all x's. When this happiness we say that $y(x)$ is closed to $y_1(x)$ in the sense of zero order proximity

Similarly we can extend the closeness of the curve $y=y(x)$ and $y=y_1(x)$ Such that both $(y(x)-y_1(x))$ and $(y'(x)-y_1'(x))$ as small for all values of x. Then we say that this two curves are closed in the sense of first order proximity.

In general the curve $y=y(x)$ and $y=y_1(x)$ are said to be closed in the sense of n^{th} order proximity. If $|y(x)-y_1(x)|, |y'(x)-y_1'(x)| \dots n^{th}$ order $|y^n(x)-y_1^n(x)|$ are small for all values of x.

Functional :- DEF

Let $y_1, y_2, \dots y_n$ be a function of ~~Simple~~ Single variable x defining the interval $[a,b]$ and S be the set of this function.

(ie) $S = \{y_1, y_2, \dots, y_n\}$.

Then any quantity which takes a definite value corresponding to the function in S said to the functional.

Eg: (1) $\int_a^b y(x)\,dx$ is functional

(2) $\int_a^b y(x)\, y'(x)\,dx$ is also functional

Continuity of the functional : DEF.

The function $I[y(x)]$ is said to be continuity at $y = y_0(x)$ in the sense of n^{th} order proximity in given any +ve number ϵ then their exists a $\delta > 0$ such that $|I[y(x)] - I[y_0(x)]| < \epsilon$

$|y(x) - y_0(x)| < \delta, |y'(x) - y_0'(x)| < \delta \dots$

$|y^n(x) - y_0^n(x)| < \delta$.

distance between two curves :-

The distance between two curves $y = y_1(x)$ and $y = y_2(x)$ for $x_0 \le x \le x_1$ is denoted by $\rho(y_1, y_2)$ which is defined as

$$\rho(y_1, y_2) = \max_{x_0 \le x \le x_1} |y_1(x) - y_2(x)|.$$

maximum and minimum:

If a functional of variation of $y(x)$ [ie $I(y(x))$] attains the maximum or minimum on the curve $y = y_0(x)$ in the values of δ on curve closed to $y = y_0(x)$ doesnot exist $I - y_0(x)$

$$\Delta = I(y_0(x)) - I[y_0(x)] \leq 0$$

$$\Delta I \leq 0$$

$\Delta I = 0$ we say that strict is attain maximum δ on $y = y_0(x)$. In the case of minimum δ on $y = y_0(x)$ $\Delta I \geq 0$

Strong and Weak Variation : DEF

If a functional of variation of $y(x)$ attains the maximum or minimum ((ie) $I(y(x))$) attains the maximum or minimum on the curve $y = y_0(x)$ with respect to the $y = y(x)$ Such that $|y(x) - y_0(x)|$ is small than 1 maximum or minimum is Said to be

Weak:

If on the otherhand variation of $y(x)$ attains the maximum or minimum on the curve $y = y_0(x)$ with respect to the curve $y = y(x)$. In the sense of first order proximity. (ie) $|y(x) - y_0(x)|$, $|y'(x) - y_0'(x)|$ are both Small then the maximum or minimum is Said to be Weak

Linear functional :-

The functional $I[y(x)]$ is defined in the linear Space M of the function $y(x)$

This functional is said to be linear fun If it is satisfied

1. $I[y(x)] = c.\ I[y(x)]$ where c is constant
2. $I[y(x)+y_1(x)] = I[y(x)] + I[y_1(x)]$

where $y(x)$ and $y_1(x) \in M$

Derivation of Euler's Equations

To determined the function $y(x)$ for which $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx$ is stationary.

Let $y = y(x)$ be the actual function which means $I[y(x)]$ is stationary and satisfies

$y(x_1) = y_1,$
$y(x_2) = y_2$ and $\eta(x)$ be the variation in y which is zero at $x = x_1$, and $x = x_2$.

(ie) $\eta(x_1) = 0, \quad \eta(x_2) = 0$

replacing y by $y + \xi[\eta(x)]$ in x

It is enough to find Stationary value of $I(\xi)$ when $\xi = 0$

The neccessary condition for the Stationary value for $I(\xi)$ such that

$$\frac{dI\xi}{d\xi} = 0 \quad \text{——— (A)}$$

Let F_{ξ} be $F(x, y + \xi\eta, y' + \xi\eta')$

$\therefore F_{\xi}$ becomes F when $\xi = 0$

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

which is known as Euler's equation (or)

Euler's lagrange equation.

Problems:

Find the estimate of functional

(1) $I[y(x)] = \int_{x_0}^{x_1} (y^2 + y'^2 - 2y\sin x)\,dx$.

Soln: Given that

$$I[y(x)] = \int_{x_0}^{x_1} (y^2 + y'^2 - 2y\sin x)\,dx \quad \text{--- (1)}$$

W.K.T

Euler's equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$I[y(x)] = \int_{x_1}^{x_1} F.(x, y, y')\,dx \quad \text{--- (2)}$$

Comparing (1) & (2)

$$F = y^2 + y'^2 - 2y\sin x$$

$$\frac{\partial F}{\partial y} = 2y - 2\sin x$$

$$\frac{\partial F}{\partial y'} = 2y'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$$

$$2y - 2\sin x - 2y'' = 0$$

$$-y'' + y - \sin x = 0$$

$$y'' - y + \sin x = 0$$

$$y'' - y = -\sin x$$

The given equation can be written as

$$(D^2 - 1)y = -\sin x$$

The auxiliary equation is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

The CF is $y = Ae^{x} + Be^{-x}$

To find Particular integral

$$PI = \frac{1}{D^2 - 1}(-\sin x)$$

$$= \frac{1}{-1-1}(-\sin x)$$

$$= \frac{\sin x}{2}$$

$\therefore$ The solution is $y = CF + PI$

$$y = Ae^{x} + Be^{-x} + \frac{\sin x}{2}$$

2. Test for estimate for the function

$$I[y(x)] = \int_0^{\pi/2} (y'^2 - y^2)\,dx, \quad y(0) = 0, \quad y(\pi/2) = 1.$$

Soln:

Given that

$$I[y(x)] = \int_0^{\pi/2} (y'^2 - y^2)\,dx \quad \text{--- (1)}$$

Euler's equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$I[y(x)] = \int_0^{\pi/2} (y'^2 - y^2)\,dx :$$

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx \quad \text{--- (2)}$$

Comparing (1) & (2)

$F = y'^2 - y^2$

$\cos \pi/2 = 0$

$\sin \pi/2 = 1$

$$\frac{\partial F}{\partial y} = -2y \quad \Big| \quad \frac{\partial F}{\partial y'} = 2y'.$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$$

$$-2y - 2y'' = 0$$

$$-y'' - y = 0$$

$$y'' + y = 0$$

The given equation can be written as

$$D^2+1=0$$

The auxiliary equation is

$$m^2+1=0$$

$$m^2=-1$$

$$m=\pm i$$

$\therefore$ CF is $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$$y = A\cos x + B\sin x.$$

To apply the initial conditions:-

$$y(0)=0$$

$$y = A\cos 0 + B\sin 0$$

$$y = A$$

$$A=0$$

$$y(\pi/2) = A\cos \pi/2 + B\sin \pi/2$$

$$1 = B \Rightarrow \boxed{B=1} \quad \therefore y = \sin x$$

3. Test for estimate for the function

$$I[y(x)] = \int_0^1 (xy+y^2-2y^2y')\,dx, \quad y(0)=1,\ y(1)=2$$

Soln:

Given that $I[y(x)] = \int_0^1 (xy+y^2-2y^2y')\,dx$ — ①

Euler equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

$$\therefore I[y(x)] = \int_{x_1}^{x_2} F(x,y,y')\,dx \quad — ②$$

comparing ① & ②,

$$F = xy+y^2-2y^2y'$$

$$\frac{\partial F}{\partial y} = x+2y-4yy'$$

$$\frac{\partial F}{\partial y'} = -2y^2$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = -4yy'.$$

$$x+2y-4yy'+4yy'=0$$

$$x+2y=0$$

$$2y=-x$$

$$y=-x/2$$

Now, $y(0)=1 \Rightarrow y(0)=-0/2$

$$1 \neq 0$$

$$y(1)=2$$

$$y(1)=-1/2$$

$$2 \neq 1/2$$

clearly this exteonal cannot satisfies the conditions. $y(0)=1$, & $y(1)=2$.

4 Show that the St. line is the Shortest distance b/w the point in the plane.

Soln:

$$s=\int_{x_1}^{x_2}\sqrt{(1+y'^2)}\,dx \quad \text{--- (1)}$$

Euler's equation is

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

$$I[y(x)]=\int_{x_1}^{x_2}F(x,y,y')dx \quad \text{--- (2)}$$

Comparing (1) & (2),

$$F=\sqrt{1+y'^2}=(1+y'^2)^{1/2}$$

$$\frac{\partial F}{\partial y'}=\frac{1}{2}(1+y'^2)^{-1/2}\,2y'=\frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=\frac{(1+y'^2)^{1/2}y''-y'(1/2)(1+y'^2)^{-1/2}\cdot 2y'y''}{1+y'^2}$$

$$=\frac{y''\sqrt{1+y'^2}-y'^2\dfrac{1}{(1+y'^2)^{1/2}}y''}{1+y'^2}$$

$$=\frac{y''(1+y'^2)-y'^2y''}{\sqrt{1+y'^2}\,.(1+y'^2)}$$

$$=\frac{y''+y'^2y''-y'^2y''}{(1+y'^2)^{3/2}}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{y''}{(1+y'^2)^{3/2}}$$

Sub in Euler's equation.

$$0 - \frac{y''}{(1+y'^2)^{3/2}} = 0$$

$$y'' = 0$$

$\int$ ing w.r.to $x \Rightarrow y' = c_1$

$$\boxed{y = c_1 x + c_2.}$$

5 $I[y(x)] = \int_0^1 \left(\left(\frac{dy}{dx}\right)^2 + 12xy\right)dx. \quad y(0)=0, \; y(1)=1$

Soln:

Given that $I(y(x)) = \int_0^1 \left(\left(\frac{dy}{dx}\right)^2 + 12xy\right)dx.$

Euler's equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

$$F = (y')^2 + 12xy$$

$$\frac{\partial F}{\partial y} = 12x$$

$$\frac{\partial F}{\partial y'} = 2y'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''.$$

Euler's equation becomes,

$$12x - 2y'' = 0$$

$$6x - y'' = 0$$

$$y'' = 6x$$

$$y' = \frac{6x^2}{2} + c_1$$

$$y = \frac{6x^3}{6} + c_1 x + c_2$$

$$y = x^3 + c_1 x + c_2.$$

$\therefore$ The Solution is $y = x^3 + c_1 x + c_2.$

$y(0) = c_2$

$0 = B$

$y(1) = c_1 + 1$

$c_1 = -1 + 1$

$$\boxed{c_1 = 0}$$

$\therefore$ The Solution is $y = x^3$.

Variational problem involving several unknown function.

1 Find the extremal of the function:-

$$v[y(x), z(x)] = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\,dx,$$

$y(0)=0,\ y(\pi/2)=-1,\quad z(0)=0,\ z(\pi/2)=1.$

Soln:

Here, $F(x, y, z, y', z')$ and

Euler's equation becomes,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$$

$$F = y'^2 + z'^2 + 2yz$$

$$\frac{\partial F}{\partial y} = 2z, \quad \frac{\partial F}{\partial y'} = 2y'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y'' \Rightarrow 2z - 2y'' = 0$$

$$z - y'' = 0$$

$$D^2y - z = 0 \quad \text{—— (1)}$$

$$\frac{\partial F}{\partial z} = 2y$$

$$\frac{\partial F}{\partial z'} = 2z'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 2z'' \Rightarrow 2y - 2z'' = 0$$

$$y - z'' = 0$$

$$z'' - y = 0$$

$$D^2 z - y = 0 \quad \text{—— (2)}$$

Multiply equation (1) by D^2

$$D^4 y - D^2 z = 0$$

$$-y + D^2 z = 0$$

$$D^4 y - y = 0$$

The Auxiliary equation is

$$m^4 - 1 = 0$$

$$(m^2+1)(m^2-1) = 0$$

$$m^2 = -1, \quad m^2 = 1$$

$$m = \pm i, \quad m = \pm 1$$

$y = c\cos x + D\sin x$ and $y = Ae^{x} + Be^{-x}$

$y = Ae^{x} + Be^{-x} + c\cos x + D\sin x$ — (3)

$y' = Ae^{x} - Be^{-x} - c\sin x + D\cos x$

$y'' = Ae^{x} + Be^{-x} - c\cos x - D\sin x$

$\therefore y'' = z = Ae^{x} + Be^{-x} - c\cos x - D\sin x$ $[\because z = y'']$.

$y(0) = Ae^{0} + Be^{-0} + c\cos 0 + D\sin 0$

$0 = A + B + C$ — (4)

$y(\pi/2) = Ae^{\pi/2} + Be^{-\pi/2} + c\cos\pi/2 + D\sin\pi/2$

$-1 = Ae^{\pi/2} + Be^{-\pi/2} + D$ — (5)

$z(0) = Ae^{0} + Be^{0} - c\cos 0 - D\sin 0$

$\Rightarrow A + B - C = 0$ — (6)

$z(\pi/2) = Ae^{\pi/2} + Be^{\pi/2} - c\cos\pi/2 - D\sin\pi/2$

$\Rightarrow Ae^{\pi/2} + Be^{-\pi/2} - D = 1 \longrightarrow$ (7).

Adding (4) & (6)

$$A + B + \cancel{C} = 0$$
$$A + B - \cancel{C} = 0$$
$$2A + 2B = 0$$
$$A + B = 0$$
$$\boxed{A = -B}$$

Adding (5) & (7)

$$Ae^{\pi/2} + Be^{-\pi/2} + \cancel{D} = -1$$
$$Ae^{\pi/2} + Be^{-\pi/2} - \cancel{D} = 1$$
$$2Ae^{\pi/2} + 2Be^{-\pi/2} = 0$$
$$Ae^{\pi/2} + Be^{-\pi/2} = 0$$
$$-Be^{\pi/2} + Be^{-\pi/2} = 0$$
$$B(e^{-\pi/2} - e^{\pi/2}) = 0$$
$$\boxed{B = 0}$$
$$A = -B$$
$$\boxed{A = 0}$$

Sub in (4) we get $0 + 0 + C = 0 \Rightarrow \boxed{C = 0}$

Sub in (5) we get $0+0+D=-1$

$$\boxed{D=-1}$$

Sub in (3) we get

$$y=-\sin x$$

$$z=\sin x$$

2. $\int_{x_0}^{x_1}(2yz-2y^2+y'^2-z'^2)\,dx$

Soln: Given that $I[y(x)]=\int_{x_0}^{x_1}(2yz-2y^2+y'^2-z'^2)\,dx$

$$I[y(x)]=\int_{x_0}^{x_2}F(x,y,z,y',z')\,dx \quad \text{—— (2)}$$

Here $F=2yz-2y^2+y'^2-z'^2$

$$\frac{\partial F}{\partial y}=2z-4y$$

$$\frac{\partial F}{\partial y'}=2y'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=2y''.$$

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

$$2z-4y-2y''=0$$

$$z-2y-y''=0$$

$$-y''-2y+z=0$$

$$y''+2y-z=0$$

$$D^2y+2y-z=0 \quad \text{—— (1)}$$

$$\frac{\partial F}{\partial z}=2y,\quad \frac{\partial F}{\partial z'}=-2z'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)=-2z''.$$

$$\frac{\partial F}{\partial z}-\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right)=0$$

$$2y+2z''=0$$

$$z''+y=0$$

$$D^2z+y=0 \quad \text{—— (2)}$$

Multiply equation (1) by D^2

$$D^4 y + D^2 2y - \cancel{D^2 z} = 0$$
$$y + 0 + \cancel{D^2 z} = 0$$

$$D^4 y + D^2 2y + y = 0$$
$$D^4 + 2D^2 + 1 = 0$$
$$(D^2+1)(D^2+1) = 0$$
$$D = \pm i, \quad D = \pm i$$
$$y = A\cos x + B\sin x, \qquad y = C\cos x + D\sin x$$

$\therefore$ The Solution is

$$y = A\cos x + B\sin x + C\cos x + D\sin x$$

3 $\int_0^1 (y'^2 + z'^2 + 2y)\,dx$, $\quad y(0)=1, \; y(1) = 3/2, \; z(0)=0, \; z(1)=1.$

Soln: Given that $I[y(x)] = \int_0^1 (y'^2 + z'^2 + 2y)\,dx$

$$F = y'^2 + z'^2 + 2y$$

$\frac{\partial F}{\partial y} = 2$

$\frac{\partial F}{\partial y'} = 2y'$

$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$

$2 - 2y'' = 0$

$y'' = 1$

$D^2 y = 0$ —— (1)

$m = 0$

$y = Ax + B$

$y(0) = B$

$B = 1$

$y(1) = A(1) + B$

$3/2 = A + 1$

$A = 3/2 - 1$

$A = 1/2$

$\boxed{y = x/2 + 1}$

$\frac{\partial F}{\partial z} = 0$

$\frac{\partial F}{\partial z'} = 2z'$

$\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 2z''$

$0 - 2z'' = 0$

$2D^2 z = 0$ —— (2)

$2m^2 = 0$

$m^2 = 0$

$\boxed{m=0}$

$z = cx + D$

$z(0) = D$

$D = 0$

$z(1) = C + 0$

$\boxed{C = 1}$

$\boxed{z = x}$

The Solution is $\quad y = x/2 + 1.$

$z = x.$

Variational Problems involving Several independent variables:-

consider the double integral

$$I = \iint_R F(x, y, z, z_x, z_y)\, dx dy$$

∴ The Euler's equation reduces to

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

This is called Euler Ostrogradsky equation

For example :-

In case of functional

$$I = \iint_R F(x, y, u, u_x\ u_y, u_{xy}, u_{yy})$$

We get the equation of extremal

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right)$$

$$+ \frac{\partial^2}{\partial x \partial y} \qquad + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{yy}}\right) = 0$$

(1) Find the Euler's ostrogradsky equation for

$$I[u(x,y)] = \iint_D \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial y}\right)^2 dx dy$$

Soln:-

Here $F = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$

$$F = u_x^2 + u_y^2$$

Euler's equation is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

$\frac{\partial F}{\partial u} = 0 \Rightarrow \frac{\partial F}{\partial u_x} = 2u_x$ | $\frac{\partial F}{\partial u_y} = 2u_y$

$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) = 2u_{xx}$ | $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 2u_{yy}$.

$$0 - 2u_{xx} - 2u_{yy} = 0$$

$$2u_{xx} + 2u_{yy} = 0$$

$$u_{xx} + u_{yy} = 0.$$

2 Find the extremal of the function

$$I[z(x,y)] = \iint_D \left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - 2z f(x,y)\right] dx\,dy$$

Soln: Given that

$$I[z(x,y)] = \iint_D \left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - 2z f(x,y)\right] dx\,dy$$

Here $F = \left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - 2z f(x,y)$

$$F = z_{xx}^2 + z_{yy}^2 + 2z_{xy}^2 - 2z f(x,y)$$

$$\frac{\partial F}{\partial z} = -2f(x,y)$$

$\frac{\partial F}{\partial z_x} = 0$ | $\frac{\partial F}{\partial z_y} = 0$

$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) = 0$ | $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$

$\frac{\partial F}{\partial z_{xx}} = 2z_{xx}$ | $\frac{\partial F}{\partial z_{yy}} = 2z_{yy}$

$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\right) = 2z_{xxx}$ | $\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial z_{yy}}\right) = 2z_{yyyy}$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial z_{xx}}\right) = 2z_{xxxx}$$

$$\frac{\partial F}{\partial z_{xy}} = 4z_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\right) = 4z_{xxy}$$

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial z_{xy}}\right) = 4z_{xxyy}$$

Euler's Equation is

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial z_{xx}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial z_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial z_{yy}}\right) = 0$$

$$-2f(x,y) - 0 - 0 + 2z_{xxxx} + 4z_{xxyy} + 2z_{yyyy} = 0$$

$$z_{xxxx} + 2z_{xxyy} + z_{yyyy} - f(x,y) = 0$$

$$f(x,y) = z_{xxxx} + 2z_{xxyy} + z_{yyyy}$$

If $f(x,y)=0$ then the equation becomes bi-harmonic equation

(3) $$I = \iint_R \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2z\, f(x,y)\right] dx\,dy$$

Soln: Here $F = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2z f(x,y)$

$$F = z_x^2 + z_y^2 + 2z f(x,y)$$

Euler Ostrogradsky equation is

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

$$\frac{\partial F}{\partial z} = 2 f(x,y)$$

$$\frac{\partial F}{\partial z_x} = 2z_x \qquad \frac{\partial F}{\partial z_y} = 2z_y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) = 2z_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 2z_{yy}$$

$$2 f(x,y) - 2z_{xx} - 2z_{yy} = 0$$

$$z_{xx} + z_{yy} = f(x,y)$$

$$f(x,y) = z_{xx} + z_{yy}$$

Functional depends on Higher Order Derivatives

consider the functional

$$I[y(x)] = \int_{x_0}^{x_1} F(x, y, y', y'')\, dx.$$

$\therefore$ The Euler equation becomes

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

Similarly, In general

$$I[y(x)] = \int_{x_0}^{x_1} F(x, y, y', y'', \ldots y^n)\, dx$$

$\therefore$ The Euler equation becomes

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial y'''}\right) + \cdots$$

$$+ (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right) = 0.$$

Problem:

1. Find the extremal of the functional

$$I[y(x)] = \int_{-a}^{a} \left(\tfrac{1}{2}\mu y''^2 + \rho y\right) dx \quad \text{subject to}$$

$$y(-a)=0, \quad y'(-a)=0, \quad y(a)=0, \quad y'(a)=0$$

Soln: Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

$$F = \tfrac{1}{2}\mu y''^2 + \rho y$$

$$\frac{\partial F}{\partial y} = \rho \qquad \frac{\partial F}{\partial y''} = \tfrac{1}{\cancel{2}}\mu \cancel{2} y'' = \mu y''$$

$$\frac{\partial F}{\partial y'} = 0 \qquad \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) = \mu y''' \Rightarrow \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = \mu y^{IV}.$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

$$\rho + y^{IV}.\mu = 0$$

$$\mu y^{IV} = -\rho$$

$$y^{IV} = -\frac{\rho}{\mu}$$

$$y''' = -\frac{\rho}{\mu}x + c_1$$

$$y'' = -\frac{\rho}{\mu}\frac{x^2}{2} + c_1 x + c_2$$

$$y' = -\frac{\rho}{\mu}\frac{x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3$$

$$\therefore y = -\frac{\rho}{\mu}\frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4$$

$$\Rightarrow y = -\frac{\rho}{\mu}\frac{x^4}{24} + Ax^3 + Bx^2 + Cx + D \quad \text{—— (1)}$$

Now differentiate we get

$$y' = -\frac{\rho}{\mu}\frac{x^3}{\cancel{24}\,6} + 3Ax^2 + 2Bx + C \quad — (2)$$

$$y(-a) = -\frac{\rho}{\mu}\frac{(-a)^4}{24} + A(-a)^3 + B(-a)^2 + C(-a) + D$$

$$\Rightarrow -\frac{\rho}{\mu}\frac{a^4}{24} - Aa^3 + Ba^2 - Ca + D \quad — (3)$$

$$y'(-a) = -\frac{\rho}{\mu}\frac{(-a)^3}{6} + 3A(-a)^2 + 2B(-a) + C$$

$$\Rightarrow -\frac{\rho}{\mu}\left(-\frac{a^3}{6}\right) + 3Aa^2 - 2Ba + C = 0 \quad — (4)$$

$$y(a) = -\frac{\rho}{\mu}\frac{a^4}{24} + Aa^3 + Ba^2 + Ca + D = 0 \quad — (5)$$

$$y'(a) = -\frac{\rho}{\mu}\frac{a^3}{6} + 3Aa^2 + 2Ba + C = 0 \quad — (6)$$

Adding (3) & (5) $\Rightarrow$

$$-\frac{\rho}{\mu}\frac{a^4}{24} \cancel{-Aa^3} + Ba^2 \cancel{-Ca} + D = 0$$

$$-\frac{\rho}{\mu}\frac{a^4}{24} \cancel{+Aa^3} + Ba^2 \cancel{-Ca} + D = 0$$

$$\Rightarrow -2\frac{\rho}{\mu}\frac{a^4}{24} + 2Ba^2 + 2D = 0$$

$$-\frac{\rho}{\mu}\frac{a^4}{24} + Ba^2 + D = 0 \Rightarrow D = \frac{\rho}{\mu}\frac{a^4}{24} - Ba^2 \quad — (7)$$

Adding (4), (6)

$$\cancel{\frac{\rho}{\mu}\frac{a^3}{6}} + 3Aa^2 \cancel{-2Ba} + C = 0$$

$$\cancel{-\frac{\rho}{\mu}\frac{a^3}{6}} + 3Aa^2 \cancel{+2Ba} + C = 0$$

$$6Aa^2 + 2C = 0$$

$$3Aa^2 + C = 0 \Rightarrow \boxed{C = -3Aa^2}$$

Sub $C = -3Aa^2$ in eqn (4)

$$\frac{\rho}{\mu}\left(\frac{a^3}{6}\right) + \cancel{3Aa^2} - 2Ba \cancel{-3Aa^2} = 0$$

$$\Rightarrow \frac{\rho}{\mu}\frac{a^3}{6} - 2Ba = 0 \Rightarrow \frac{\rho}{\mu}\frac{a^3}{6} \times \frac{1}{2a} = B \Rightarrow B = \frac{\rho}{\mu}\frac{a^2}{12}$$

$$D = \frac{\rho}{\mu}\frac{a^4}{24} - \frac{\rho}{\mu}\frac{a^2}{12} \times a^2$$

$$D = \frac{\rho}{\mu}\frac{a^4}{24} - \frac{\rho}{\mu}\frac{a^4}{12} = \frac{\rho}{\mu}\frac{a^4}{12}\left(\frac{1}{2} - 1\right)$$

$$\boxed{D = -\frac{\rho}{\mu}\frac{a^4}{24}}$$

Sub in 'D' values in ③

$$-\frac{\rho}{\mu}\frac{a^4}{24} - Aa^3 + \frac{\rho}{\mu}\frac{a^2}{12} \times a^2 + 3Aa^3 + -\frac{\rho}{\mu}\frac{a^4}{24} = 0$$

$$\Rightarrow -2\frac{\rho}{\mu}\frac{a^4}{24} + \frac{\rho}{\mu}\frac{a^4}{12} + 2Aa^3 = 0$$

$$2Aa^3 = 0$$

$$\boxed{A = 0}$$

$$C = 0$$

$$\boxed{\therefore y = -\frac{\rho}{\mu}\frac{x^4}{24} + x^2\frac{\rho}{\mu}\frac{a^2}{12} - \frac{\rho}{\mu}\frac{a^4}{24}}$$

Variational Problem in Parametric form:-

Consider the function

$$I[x(t), y(t)] = \int_{t_0}^{t_1} F(t, x, y, \dot{x}, \dot{y})\,dt$$

the weirstrass form of Euler equation

$$\frac{1}{r} = \frac{\phi_{x\dot{y}} - \phi_{y\dot{x}}}{\phi_1(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

$$\text{where } \phi_1 = -\frac{\phi_{\dot{x}\dot{y}}}{\dot{x}\dot{y}}$$

Problem:

1. Consider the variational Problem to foll. extremal

$$I[x(t), y(t)] = \int_{t_0}^{t_1} \left[(\dot{x}^2 + \dot{y}^2)^{1/2} + a^2(x\dot{y} - y\dot{x})\right] dx$$

Soln:

Here $F = (\dot{x}^2 + \dot{y}^2)^{1/2} + a^2(x\dot{y} - y\dot{x})$

The Weirstrass form of Euler equation

$$\frac{1}{r} = \frac{\phi_{x\dot{y}} - \phi_{y\dot{x}}}{F_1(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

$$\text{where } F_1 = \frac{-F_{\dot{x}\dot{y}}}{\dot{x}\dot{y}}$$

$$\frac{\partial F}{\partial x} = a^2\dot{y} \Rightarrow \frac{\partial^2 F}{\partial x\,\partial \dot{y}} = a^2.$$

$$\frac{\partial F}{\partial y} = -a^2\dot{x}$$

$$\frac{\partial^2 F}{\partial y \partial \dot{x}} = -a^2$$

$$\frac{\partial F}{\partial \dot{x}} = \frac{1}{2}(\dot{x}^2+\dot{y}^2)^{3/2} \cdot 2\dot{x} - a^2 y$$

$$\frac{\partial^2 F}{\partial \dot{x} \partial \dot{y}} = -1/2 (\dot{x}^2+\dot{y}^2)^{-3/2} \dot{x}\dot{y}\dot{x}$$

$$= \frac{-\dot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}$$

$$F_1 = \frac{\dot{x}\dot{y}}{\dot{x}\dot{y}(\dot{x}^2+\dot{y}^2)^{3/2}}$$

$$1/r = \frac{a^2+a^2}{\frac{1}{\cancel{(\dot{x}^2+\dot{y}^2)^{3/2}}}} \cdot \cancel{(\dot{x}^2+\dot{y}^2)^{3/2}}$$

$$1/r = 2a^2$$

Hamilton's Principle :-

Consider a Particle of mass 'm' moving in a force field if the position vector of the particle with respect to the fixed origin is denoted by 'r'.

Then by newton's law motion the path of the Particle is

$$m \times \frac{d^2 r}{dt^2} = f \rightarrow (1)$$

Where f is the force acting on the Particle

Now, consider any other path $r+\delta r$ on the Provise that the true path and varied Path coincide at two distinct instants and $t=t_1$ and $t=t_2$

This demands that δr at $t_1=0$ and δr at $t_2=0$.

at any intermediate time 't'.

We examine the true path and varied path '$r + \delta r$'

Taking the dot product of variation in eqn ①

and integrating their result with respect to the time

over the $[t_1, t_2 = 0]$.

We get $m \times \frac{d^2 r}{dt^2} - f = 0$ —— ②

$$\int_{t_1}^{t_2} \left[m \frac{d^2 r}{dt^2} \delta r - f \delta r \right] dt = 0$$

$$\int_{t_1}^{t_2} m \frac{d^2 r}{dt^2} \delta r dt - \int_{t_1}^{t_2} f \delta r dt = 0 \quad \text{—— ③}$$

Consider the 1^{st} integral of eqn ③

$$\int_{t_1}^{t_2} m \frac{d^2 r}{dt^2} \delta r dt = \int_{t_1}^{t_2} m \frac{d}{dt} \frac{dr}{dt} \delta r dt$$

$$= \int_{t_1}^{t_2} m \, d\left(\frac{dr}{dt}\right) \delta r$$

$$= m \int_{t_1}^{t_2} d\left(\frac{dr}{dt}\right) \delta r$$

$u = \delta r \qquad dv = d\left(\frac{dr}{dt}\right)$

$\frac{du}{dt} = \frac{d}{dt}(\delta r) \qquad v = \frac{dr}{dt}$

$du = \frac{d}{dt}(\delta r) dt$

$$\int_{t_1}^{t_2} m \frac{d^2 r}{dt^2} \delta r dt = m \left[\left[\delta r \frac{dr}{dt} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{dr}{dt} \frac{d}{dt}(\delta r) \right]$$

$$= 0 - m \int_{t_1}^{t_2} \frac{dr}{dt} \delta\left(\frac{dr}{dt}\right) dt \quad (\text{by } ②)$$

$$= -m \int_{t_1}^{t_2} \delta\left(\frac{dr}{dt}\right)^2 dt$$

$$= -\delta \int_{t_1}^{t_2} \frac{m}{2} \left(\frac{dr}{dt}\right)^2 dt$$

$$= -\delta \int_{t_1}^{t_2} T \, dt.$$

where T is the Kinetic Energy $\frac{1}{2} m \left(\frac{dr}{dt}\right)^2$ of the Particle

Sub in eqn (3)

$$\int_{t_1}^{t_2} -\delta T\, dt - \int_{t_1}^{t_2} f\, \delta r\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \delta T\, dt + \int_{t_1}^{t_2} f\, \delta r\, dt = 0$$

$$\int_{t_1}^{t_2} (\delta r + f\, \delta r)\, dt = 0 \quad \text{--- (4)}$$

This is Hamilton Principle of Single Particle.

Now, consider the case when the force field f having components (X, Y, Z) is Conservative.

Which implies that $f = X_i + Y_j + Z_k$.

$$dr = dx_i + dy_j + dz_k$$

Which implies $f\, dr = x\,dx + y\,dy + z\,dz$ is the differential of the single valued function

$$\phi = (X, Y, Z)$$

This function is called force potential and its negative say 'v' is the potential energy of the particle.

Thus $f\, \delta r = \delta\phi = -\delta V$

Sub in (4) $\Rightarrow \int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0$

$$\delta \int_{t_1}^{t_2} (t - v)\, dt = 0.$$

Which is Hamilton Principle.

To derive the equation of Vibration of bar:

A displacement from equlibrium of the position $U(x, t)$ will be a function of time 't'.

The kinetic energy of the bar of the 'l'.

$$\therefore T = \frac{1}{2} \int_0^l \rho \left(\frac{\partial y}{\partial t}\right)^2 dx.$$

We assume that the bar is sligetly extensibly the potential energy of a ______ ball with a constant curvature is — to the square of the curvature.

Thus the differential 'dv' of the potential energy of the bar is

(ie) $dv \propto \frac{1}{\rho^2}$

where $\rho = \frac{\left\{1+\left(\frac{\partial u}{\partial x}\right)^2\right\}^{3/2}}{\frac{\partial^2 u}{\partial x^2}}$

$$dv \propto \frac{1}{\left[\frac{\left\{1+\left(\frac{\partial u}{\partial x}\right)^2\right\}^{3/2}}{\frac{\partial^2 u}{\partial x^2}}\right]^2}$$

$$dv \propto \left[\frac{\frac{\partial^2 u}{\partial x^2}}{\left\{1+\left(\frac{\partial u}{\partial x}\right)^2\right\}^{3/2}}\right]^2$$

$$dv = \frac{1}{2} k \left[\frac{\frac{\partial^2 u}{\partial x^2}}{\left\{1+\left(\frac{\partial u}{\partial x}\right)^2\right\}^{3/2}}\right]^2$$

$$v = \frac{1}{2} k \int_0 \left[\frac{\frac{\partial^2 u}{\partial x^2}}{\left[1+\left(\frac{\partial u}{\partial x}\right)^2\right]^{3/2}}\right]^2 . dx.$$

According to assumption of slide extensiblity the derivation of the bar from the equilibrium position are small and that term $\left(\frac{\partial u}{\partial x}\right)^2$ may be ignore.

$$V = \frac{1}{2} k \int_0^l \left(\frac{\partial^2 u}{\partial x^2}\right) dx.$$

Now, by Hamilton's principle.

$$\int_{t_1}^{t_2} \int_0^l \left\{\frac{1}{2} \rho \left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2} k \left(\frac{\partial^2 u}{\partial x^2}\right)^2\right\} dx\, dt = 0$$

will be a extremum for a fixed terminal times t_1 and t_2..

The Euler Ostrogradsky equation

Then gives $\frac{\partial}{\partial t}\left(\rho \frac{\partial U}{\partial t}\right) + \frac{\partial^2}{\partial x^2} k\left(\frac{\partial^2 U}{\partial x^2}\right) = 0$

$$F(x,t,U,U',U'') = \tfrac{1}{2}\rho\left(\frac{\partial U}{\partial t}\right)^2 - \tfrac{1}{2}k\left(\frac{\partial^2 U}{\partial x^2}\right)^2$$

$$\frac{\partial F}{\partial U} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial U_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial U_t}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial U_{xx}}\right)$$
$$+ \frac{\partial^2}{\partial t^2}\left(\frac{\partial F}{\partial U_{tt}}\right) + \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial F}{\partial U_{xt}}\right) = 0$$

$$\frac{\partial F}{\partial U} = 0 \qquad \frac{\partial F}{\partial U_t} = \rho u_t$$

$$\frac{\partial F}{\partial U_x} = 0 \qquad \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right) = \rho u_{tt}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial U_x}\right) = 0$$

$$\frac{\partial F}{\partial u_{xx}} = -k u_{xx} \qquad \frac{\partial F}{\partial u_{tt}} = 0$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) = -k u_{xxxx} \qquad \frac{\partial^2}{\partial t^2}\left(\frac{\partial F}{\partial u_{tt}}\right) = 0$$

$$\frac{\partial F}{\partial u_{xt}} = 0$$

$$\frac{\partial^2}{\partial x \partial t}\left(\frac{\partial F}{\partial u_{xt}}\right) = 0$$

$$-\rho u_{tt} - k u_{xxxx} = 0 \Rightarrow \rho \frac{\partial^2 u}{\partial t^2} +$$

$$\frac{\partial}{\partial t}\rho\left(\frac{\partial u}{\partial t}\right) + \frac{\partial^2}{\partial x^2} k \frac{\partial^2 u}{\partial x^2} = 0$$

Which is the equation for displacement $u(x,t)$.

III, we can derived the one-dimensional wave equation for the displacement $u(x,t)$ a highly flexible almost in entensially string from equilibrium position is

$$\frac{\partial^2 u}{\partial t^2} - \frac{t}{\rho}\left(\frac{\partial^2 u}{\partial x^2}\right) = 0$$

where t and ρ denotes the tension — line density of the string there is a important generalization of the above.

When the string is acted by a distributed linear restored force towards the equilibrium position.

This leads to adding a term $-ku$ to the R.H.S of the above equation.

The New equation is known as the order equation

Which its general form is given by

$$\frac{\rho}{T}\frac{\partial^2 U}{\partial t^2} = \nabla^2 U - kU$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 U - kU \quad \therefore c^2 = \frac{T}{\rho}.$$

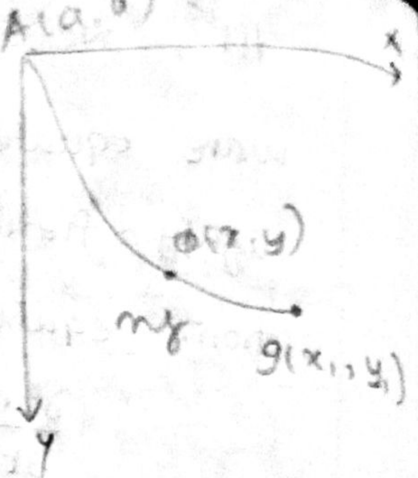
Branchistochrone Problem:

Find the curve Joining given period A and B which is traversa by a particle moving under gravity from A to B in the Shortest time.

This is known as the brachistochrone problem.

Soln:

Fix the origin at A with x axis horizontal and y axis vertically downward



The speed of the particle $\frac{ds}{dt}$ is given by $\sqrt{2gy}$.

(ie) $\frac{ds}{dt} = \sqrt{2gy}$ where g being acceleration due to gravity.

Thus the time taken by the particle in moving from $A(0,0)$ to $B(x_1, y_1)$

$$t(y(x)) = \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx, \quad y(0), \ y(x_1) = y_1$$

Here $F = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$.

Since the integral is independent

The 1st integral of Euler's equation

$$F - y' \frac{\partial F}{\partial y'} = c_1 \quad \text{—} ①$$

Here $F = \frac{\sqrt{1+y'^2}}{\sqrt{y}} = \frac{1}{\sqrt{y}}(1+y'^2)^{1/2}$

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y}} \frac{1}{2} (1+y'^2)^{1/2-1} \cdot 2y'$$

$$= \frac{(1+y'^2)^{-1/2} \cdot 2y'}{2\sqrt{y}}$$

$$= \frac{y'}{\sqrt{y} \cdot \sqrt{1+y'^2}}$$

Sub $\frac{\partial F}{\partial y'}$ value in ①

$$\Rightarrow \frac{\sqrt{1+y'^2}}{\sqrt{y}} - y' \cdot \frac{y'}{\sqrt{y}\sqrt{1+y'^2}} = c_1$$

$$\Rightarrow \frac{\sqrt{1+y'^2}\sqrt{1+y'^2} - y'^2}{\sqrt{y}\sqrt{1+y'^2}} = c_1$$

$$1+\cancel{y'^2}-\cancel{y'^2} = c_1\sqrt{y(1+y'^2)}$$

$$1 = c_1\sqrt{y(1+y'^2)}$$

$$c_1 = \frac{1}{\sqrt{y(1+y'^2)}} \quad (\text{or}) \quad \frac{1}{c_1} = \sqrt{y(1+y'^2)}$$

Squaring on both Sides,

$$\left(\frac{1}{c_1}\right)^2 = y(1+y'^2)$$

$$y(1+y'^2) = c_1 \longrightarrow (2).$$

$y' = \cot t$, t being a Parameter, then eqn (2)

becomes $$y(1+\cot^2 t) = c_1$$

$$y = \frac{c_1}{1+\cot^2 t} \Rightarrow y = \frac{c_1}{\operatorname{cosec}^2 t}$$

which implies, $y = c_1 \sin^2 t$

$$\Rightarrow y = c_1\left(\frac{1-\cos 2t}{2}\right)$$

$$y = \frac{c_1}{2}(1-\cos 2t) \longrightarrow (3)$$

$$\frac{dy}{dt} = \frac{c_1}{2}(2\sin 2t) = c_1 \sin 2t$$

$$dy = c_1 \sin 2t \, dt$$

Now, $y' = \frac{dy}{dx}$

$$\Rightarrow dx = \frac{dy}{y'}$$

$$dx = \frac{c_1 \sin 2t \, dt}{\cot t} = \frac{c_1 \, 2\sin t \cos t \, dt}{\cot t}.$$

$$= \frac{c_1 \, 2\sin t \cancel{\cos t} \sin t}{\cancel{\cos t}} = 2c_1 \sin^2 t \, dt$$

$$= 2c_1\left(\frac{1-\cos 2t}{2}\right)dt$$

$$dx = c_1(1-\cos 2t)\,dt$$

$\int$ing, we get $$x = c_1\left(t - \frac{\sin 2t}{2}\right) + c_2.$$

$$x - c_2 = \frac{2c_1t - \sin 2t \cdot c_1}{2}$$

$$x - c_2 = \frac{2c_1t - c_1 \sin 2t}{2}$$

$$x - c_2 = \frac{c_1}{2}(2t - \sin 2t) \quad \text{— (4)}$$

Put $2t = t_1$ in eqn (4)

Given $y(0) = 0 \Rightarrow x = 0, y = 0$

Take $c_2 = 0$, we get

equation (4) becomes $x = \frac{c_1}{2}(t_1 - \sin t_1)$

$$x = \frac{c_1}{2}(t_1 - \sin t_1)$$

Thus, equation (3) & (4) gives the desired extremal in the parametric form

$$x = \frac{c_1}{2}(t_1 - \sin t_1)$$

$$y = \frac{c_1}{2}(1 - \cos t_1)$$

Which is the family of cycloid $\frac{c_1}{2}$ as the radius of the rolling circle.

Impact c_1 is determined by the that the cycloid passes through $B(x_1, y_1)$