Measure Theory and Integration

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Measure and outer measure - Outer measure and measurability - The Extension theorem - product measures

Chapter 12 : Sec 1, 2 and 4

Text Book

H. L. Royden - Real Analysis, Mc Millian publ, Company, New york 1993

Reference Book

1. G. de Barra - Measure Theory and Integration, Wiley Eastern Ltd, 1981

2. P. K. Jain and V. P. Gupta - Lebesgue Measure and Integration, New Age Int. (P) Ltd, New Delhi.

3. Walter Rudin - Real and complex Analysis, Tata Mc. Graw Hill publ Co Ltd, New Delhi 1966

Unit - I

Lebesgue Measure

Introduction

The Length $l(I)$ of an interval I is defined to be the difference of the end points of the interval.

Length is an example of a set function that associates an extended real number to each set in some collection of sets.

We defined the length of an open set to be sum of the length of the open interval of which it is composable.

we defined a set function "m" that assigns to each set E in some collection m of sets of real numbers, a non negative extended real number "mE" called the measure of E

satisfying the following properties

1. mE is defined for each set E of real numbers

(i.e) $m = \wp(R)$

2. For an interval I, $m'I = l(I)$

3. If $\{E_n\}$ is a sequence of disjoint sets then $m(\cup E_n) = \sum m E_n$

4. m is translation invariant

(i.e) If E is a set for which m is defined and if $E+y$ is the set $\{x+y / x \in E\}$ then $m(E+y) = m(E)$

σ- Algebra

An algebra $\mathcal{a}$ of sets is called a σ-Algebra (or) a borel field, if every union of a countable collection of sets in $\mathcal{a}$ is again in $\mathcal{a}$

(i.e) If $\{A_i\}$ is a sequence of sets then $\bigcup_{i=1}^{\infty} A_i$ must be again in $\mathcal{a}$

Outer Measure

For each set A of a numbers, consider the countable collection $\{I_n\}$ of open intervals that cover A

(i.e) Collection for which $A \subset \cup I_n$ and for each such collection consider the sum of the length of the intervals in the collection.

The outer measure m^*A of A is defined by → Greatest lower bound → Infimum

$$m^*A = \inf_{A \subset \cup I_n} \sum l(I_n)$$

where the infimum taken over all finite or countable collection of intervals $\{I_n\}$ such that $A \subset \cup I_n$ $(-1/n, 1+1/n)$ $[0\ 1] \subset (\frac{-1}{2}\ \frac{3}{2}) \cup [\ \ ) \cup (\ \ ]$

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(or) Let A be a set of real numbers, the outer measure m^*A of A is defined

$$m^*A \leq \sum l(I_n)$$

Where $l(I_n)$ denoted the length of I_n.

Notes

1. If A is a set of real number, we can cover A by a countable collection of open intervals.

2. $m^*A \geq 0, \ \forall A$

(i.e) The outer measure is always non negative.

3. Since $m^*A = \inf\limits_{A \subset \cup I_n} \sum l(I_n)$

Given $\epsilon > 0$ there exist I_n Such that $A \subset \cup I_n$ and $\sum l(I_n) < m^*A + \epsilon$

4. The outer measure m^* is called **Lebesgue** outer measure.

Supremum - least upper bound.

Results

1. If $A \subset (a,b)$ then $m^*A \le b-a$

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Proof

we know that

$$m^*A = \inf_{A \subset \cup I_n} \sum l(I_n)$$

Here $A \subset (a,b)$

$$L[(a,b)] = b-a$$

Since $m^*A \le L(I_n)$

$$m^*A \le b-a$$

Hence the proof

2. $A \subset B$ then $m^*A \le m^*B$

Proof

Let $A \subset \cup I_n$ and $B \subset \cup J_n$

Since $A \subset B$

which implies that

$$\cup I_n \subset \cup J_n$$

Now $m^*A \le \sum l(I_n)$ and $m^*B \le \sum l(J_n)$

Therefore $m^*A \le m^*B$

Hence the proof

3. Prove that $m^*(\{x\}) = 0$, x is any real.

Proof

Since $x \in R$, there exist $\epsilon > 0$

$\{x\} \subset (x-\epsilon, x+\epsilon)$

Now then

$m^*A \leq \Sigma l(I_n)$

$$m^*(\{x\}) \leq x+\epsilon-(x-\epsilon)$$
$$\leq \cancel{x}+\epsilon-\cancel{x}+\epsilon$$
$$\leq 2\epsilon \cdot \forall \epsilon > 0$$

$m^*(\{x\}) = 0$ (By result 1)

Hence the proof

4. Prove that $m^*\phi = 0$

Proof

we know that "ϕ" is a subset of any set

Let $\phi \subset \{x\}$

$$m^*\phi \leq m^*\{x\}$$

$m^*\phi = 0$ [$\because m^*$ is non-negative]

Preposition 1 :

The outer measure of an interval is its length (or)

prove that $m^*I = l(I) = b-a$

Proof

Case 1:

Let I be a finite closed interval. Say $[a,b]$

Since $I \subset (a-\epsilon, b+\epsilon)$.

we have $m^*n \le l(I_n)$

$$m^*I \le b+\epsilon-(a-\epsilon)$$

$$\le b+\epsilon-a+\epsilon$$

$$\le b-a+2\epsilon$$

$$m^*I \le b-a \longrightarrow (1) \ \forall \epsilon>0$$

Next we have to prove that

$$m^*I \ge b-a.$$

Let $I \subset \bigcup_{n=1}^{\infty} I_n$

By heine borel theorem, there exist a finite sub collection $\{I_n\}$,

Say $I_1, I_2, \ldots I_m$ which covers I

(i.e) $I \subset \bigcup_{n=1}^{m} I_n$

Also we have,

$$\sum_{n=1}^{m} l(I_n) \le \sum_{n=1}^{\infty} l(I_n)$$

Since $I = [a,b] \subset I_1 \cup I_2 \cup \ldots \cup I_m$

we have $a \in I_l$ ~~for some~~ $\cancel{1 \le l \le m}$

Let $I_l = (a_1, b_1)$

we have $a_1 < a < b_1$

If $b_1 \ge b$

Let condition

we get $a_1 < a < b < b_1$

$\therefore l(I_l) > b-a$

$b-a < b_1-a_1$

$b_1-a_1 \ge b-a$

$l(I_l) > b-a$

which implies that

$$\sum_{n=1}^{m} \lambda(I_n) > \lambda(I_1) > b-a$$

$$\sum_{n=1}^{m} \lambda(I_n) > b-a \quad \text{——} (2)$$

$b_1 \in (a, b)$

$b_1 \notin (a_1, b_1)$

Let → If $b_1 < b$ then $b_1 \in [a, b]$

But $b_1 \notin I_1$

∴ there is an interval in the collection $\{I_n\}$ distinct from (a_1, b_1)

for which $b_1 \in I_2 = (a_2, b_2)$

$a_2 < a_1 < a < b_1$

$b_1 < b_2$

Then $a_2 < b_1 < b_2$

If ~~$b_2 \leq b$~~ $b_2 \geq b$, the inequality (2) is established as

since $\sum_{n=1}^{m} \ell(I_n) \geq (b_1 - a_1) + (b_2 - a_2) = b_2 - (a_2 - b_1) - a_1 > b_2 - a_1 > b-a$

Continuing in this way

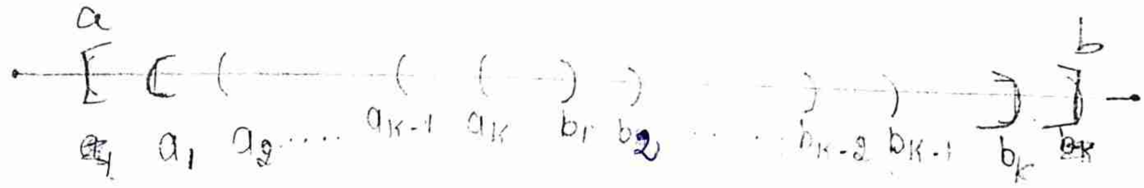
we obtain a sequence $(a_1, b_1), (a_2, b_2) \ldots (a_k, b_k)$

Since $\{I_n\}_{n=1}^{m}$ is a finite collection, this

process must terminate with some end point interval

say (a_k, b_k)

where $k \leq m$



But it terminates only if $b \in (a_k, b_k)$

Thus now

$$\sum_{i=1}^{k} \lambda(I_i) \geq b_1 - a_1 + b_2 - a_2 + \ldots\ldots + b_k - a_k$$

$$= b_k - a_k + b_{k-1} - a_{k-1} \ldots\ldots + b_3 - a_3 + b_2 - a_2 + b_1 - a_1$$

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$$\sum_{i=1}^{K} l(I_i) = b_K + (b_{K-1} - a_K) + (b_{K-2} - a_{K-1}) + \cdots + (b_2 - a_3) + (b_1 - a_2) - a_1$$

$$\sum_{i=1}^{K} l(I_i) > (b_K - a_1) \quad [\because a_i < b_{i-1} < b_i]$$

$i = 2$

$a_2 < b_1 < b_2$

$b_1 - a_2$

But $b_K > b$ and $a_1 < a$

(i.e) $a_1 < a < b < b_K$

$\Rightarrow b_K - a_1 > b - a$

$\Rightarrow \sum_{i=1}^{\infty} l(I_i) > \sum_{i=1}^{K} l(I_i) > b - a$

(i.e) $\sum_{n=1}^{\infty} l(I_n) > b - a$

$\inf \sum_{n=1}^{\infty} l(I_n) > b - a$

$m^* I \geq b - a \longrightarrow (2)$

From (1) and (2)

$m^* I = b - a$

case 2:

Let I be any finite interval

(i.e) $I = [a, b)$ (or) $(a, b]$

For, given $\epsilon > 0$ there exist $J = [a + \frac{\epsilon}{4}, b - \frac{\epsilon}{4}]$

Such that $J \subset I$

$\therefore L(J) = b - \frac{\epsilon}{4} - (a + \frac{\epsilon}{4})$

$= b - a - \frac{\epsilon}{4} - \frac{\epsilon}{4}$

$= b - a - \epsilon/2$

$$L(J) = L(I) - \frac{\epsilon}{2}$$

Now $L(J) + \frac{\epsilon}{2} > L(I) - \frac{\epsilon}{2} \quad \forall \ \epsilon > 0$

$$\Rightarrow \quad L(J) > L(I) - \frac{\epsilon}{2} - \frac{\epsilon}{2}$$

$$L(J) > L(I) - \epsilon \longrightarrow (3) \quad \forall \ \epsilon > 0$$

Since $J \in I$ $\quad L(J) < L(I)$

$$L(J) = \boxed{m^*(J) \leq m^* I} \leq m^* \bar{I} = L(\bar{I}) = L(I)$$

Where $\bar{I} = [a, b]$ $\quad L(I) = m^*(I)$

Now from (3)

$$\Rightarrow L(I) - \epsilon < L(J) \leq m^* I \leq L(I)$$

$$L(I) - \epsilon \leq m^* I \leq L(I) \quad \forall \ \epsilon > 0$$

$$\Rightarrow L(I) \leq m^* I \leq L(I)$$

$$\Rightarrow \quad m^* I = L(I) = b - a$$

(i.e) $m^* I = b - a$

Case 3 :

Let I be an infinite interval

Given any real number Δ there is a closed interval $J \subset I$ with $L(J) = \Delta$

$$\Delta = L(J) = m^* J \leq m^* I$$

$$\Delta \leq m^* I \quad \forall \ \Delta$$

we get $m^* I = \infty = L(I)$

Hence the proof

Preposition 2 : Sub. additivity property.

Let $\{A_n\}$ be a countable collection of sets of real numbers. Then $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^* A_n$

Proof

If $m^* A_n = \infty$ for some n then the inequality holds trivially

$\therefore$ The assume that $m^* A_n < \infty \quad \forall n$

Then given $\epsilon > 0$ there is a countable collection $\{I_{n,i}\}_i$ of open intervals such that $A \subset \bigcup_{i=1}^{\infty} I_{n,i}$ and

$$\sum_i l(I_{n,i}) < m^* A_n + 2^{-n} \epsilon \rightarrow (1)$$

Since countable number of countable collection is again a countable

we have the collection $\{I_{n,i}\}_{n,i} = \bigcup_n \{I_{n,i}\}_i$ is countable.

Also $\bigcup A_n \subset \bigcup_{n=1}^{\infty} \{I_{n,i}\}_i$

$$m^*(\bigcup A_n) \leq m^*\left(\bigcup_{n=1}^{\infty} \{I_{n,i}\}_i\right)$$

$$\leq \sum_{n,i}^{\infty} l(I_{n,i})$$

$$\leq \sum_n \sum_i l(I_{n,i})$$

$$< \sum_n \left[m^* A_n + 2^{-n} \epsilon\right]$$

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$$m^*(\cup A_n) \leq \sum_n m^* A_n + \sum_n 2^{-n} \epsilon \qquad \sum 2^{-n} = \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3}$$

$$\frac{a}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

$$m^*(\cup A_n) \leq \sum_n m^* A_n + \epsilon$$

ϵ cancel [illegible]

" " [illegible]

$$m^*(\cup A_n) \leq \sum_{n=1}^{\infty} m^* A_n \quad \forall \ \epsilon > 0$$

Hence the proof

Corollary 1:

If A is countable then $m^* A = 0$

Proof

Let $A = \{x_1, x_2, \ldots x_n\}$

$$A = \cup x_i$$

$$m^* A = m^*(\cup x_i)$$

$$\leq \sum_{i=1}^{\infty} m^* x_i$$

Single point [illegible] measure 0

$$m^* A \leq 0 \qquad [m^* A \geq 0]$$

Hence $m^* A = 0$

Hence the proof

Corollary 2 :

The set $[0,1]$ is uncountable

Proof

Suppose the set $[0,1]$ is countable

$\Rightarrow m^*([0,1]) = 1-0$

$m^*[0,1] = 0$ (by corollary 1)

We know that

The outer measure of an interval

is its length

(i.e) $m^* I = \lambda(I) = b-a$

Then, we have

$$m^*[0,1] = \lambda([0,1]) = 1-0 = 1$$

which is a contradiction

$\therefore [0,1]$ is uncountable.

Hence the proof

Note

1). The set of all rational numbers $\mathbb{Q}$ is countable.

$\therefore m^* \mathbb{Q} = 0$

2). The preposition 2 is called " The countable sub additivity of m^* "

clearly m^* satisfies finite subadditivity.

(i.e) $m^*\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m^* A_i$

preposition 3 :

Given any set A and any $\epsilon > 0$, there exist an open set U such that $A \subset U$ and $m^* U \leq m^* A + \epsilon$.

Proof There is a set $G \in G_\delta$ such that $A \subset G$ and $m^* A = m^* G$ [G_δ intersection of open sets

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open intervals are open set, Union of open intervals is open

Proof

Let A be a set of real numbers then there exist an open intervals I_n such that $A \subset \cup I_n$ and

$$\sum l(I_n) \le m^*A + \epsilon \rightarrow ①$$

Let $U = \cup I_n$

Then U is an open set and $A \subset U$

$$\Rightarrow m^*U \le m^*A + \epsilon$$

To find a G_δ set take $\epsilon = \frac{1}{n}$

For each "n" there exist a U_n such that $A \subset U_n$ and

$A \subset G_1$

$$m^*U_n \le m^*A + \epsilon$$

$$m^*U_n \le m^*A + \frac{1}{n} \rightarrow ②$$

G_δ

Let $G = \bigcap_n U_n$

$G = \bigcap_n U_n$, $G \subset U_n$, since $A \subset U_n$; $A \subset \cap U_n$, $A \subset G$, (i.e) $G \supset A$.

Then G is a G_δ set and $G \supset A$

Also we have

$$m^*G \le m^*U_n \quad [\because G \subset U_n]$$

$$m^*G \le m^*A + \frac{1}{n} \quad \forall n$$

$$m^*G \le m^*A \rightarrow ③$$

Since $G \supset A$

$$m^*G \ge m^*A \rightarrow ④$$

From ③ and ④, we get

$$m^*G = m^*A$$

Hence the proof

Problems

1. Prove that $m^*A = 0$ then $m^*(A \cup B) = m^*B$

Proof

Since $m^*(A \cup B) \leq m^*A + m^*B$ [triangle inequality]

Given that $m^*A = 0$

$\Rightarrow m^*(A \cup B) \leq m^*B \rightarrow (1)$

And also, $B \subset A \cup B$ (w.k that)

$\Rightarrow m^*B \leq m^*(A \cup B) \longrightarrow (2)$

From ① and ② we get

$$m^*(A \cup B) = m^*B$$

Hence the proof

2. Prove that m^* is translation invariant

means ஒரு set [illegible] ஒரு number [illegible] (or) measure [illegible]

Proof

We have to prove that

$$m^*(A+x) = m^*A \quad \forall x \in R$$

Given $\epsilon > 0$ there exist a sequence $\{I_n\}$ such that $A \subset \bigcup I_n$ and

$$\sum l(I_n) \leq m^*A + \epsilon \longrightarrow (1)$$

Since $A \subset \bigcup I_n$

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$\Rightarrow \quad A+x \subset \cup (I_n+x)$

$m^*(A+x) \leq \Sigma l(I_n+x)$

$m^*(A+x) \leq \Sigma l(I_n)$

From ①

$m^*(A+x) \leq m^*A + \epsilon$

Since ϵ is arbitrary

$m^*(A+x) \leq m^*A \longrightarrow$ ②

Then now

$A = (A+x) - x$

$A = (A+x) + y$

where $y = -x$

The above is using in ②

$\Rightarrow m^*A = m^*[(A+x)+y]$

$m^*A \leq m^*(A+x) \longrightarrow$ ③

From ② and ③

$m^*(A+x) = m^*A$

Hence the proof

3. Let A be the set of all rational numbers between 0 and 1. Let $\{I_n\}$ be the collection of open intervals covering A then $\Sigma l(I_n) \geq 1$

proof

Let $I_1, I_2, \ldots I_n$ be a finite collection of open intervals such that

$$A \subset I_1 \cup I_2 \cup \ldots \cup I_n \rightarrow \text{open intervals}$$

W.K.T The open are in closed interval

$$(i.e)\ A \subset \bigcup_{i=1}^{n} I_i$$

$$\Rightarrow [0,1] \subset \bigcup_{i=1}^{n} \bar{I}_i \rightarrow \text{closed intervals}$$

$$\Rightarrow m^*[0,1] \leq \sum_{i=1}^{n} l(\bar{I}_i)$$

$$l(0,1) = 1-0 = 1$$

$$1 \leq \sum_{i=1}^{n} l(\bar{I}_i)$$

$$(i.e)\ 1 \leq \sum l(I_n)$$

$$\Rightarrow \sum l(I_n) \geq 1$$

Hence the proof

Measurable set and Lebesgue Measure

Measurable set

A subset E of R is said to be measurable if for each subset $A(R)$ we have,

$$m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$$

where $\tilde{E} = R - E$

(i.e) $\bar{E}$ is the complement of E in R.

Note

1). Since Lebesgue outer measure m^* is the measurable set are called Lebesgue measurable set.

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2) we have

$$A = (A \cap E) \cup (A \cap \tilde{E})$$

$m^*A = m^*[(A \cap E) \cup (A \cap \tilde{E})]$

since $m^*(\cup A) \leq \sum m^*A$

$$m^*A \leq m^*(A \cap E) + m^*(A \cap \tilde{E}) \quad [\because \text{proposition 2}]$$

we always have this result

This is trivial in all place

Hence E is measurable iff for each

set $A \subset R$

$$m^*A \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

3) Since the definition of measurability is

Symmetric in E and $\tilde{E}$.

we have $\tilde{E}$ is measurable, whenever E

is measurable.

Result T.P. R is measurable

If $E = R$ we have

$$m^*A = m^*(A \cap R) + m^*(A \cap \tilde{R})$$

$$= m^*A + m^*(A \cap \phi)$$

$$= m^*A + m^*\phi$$

$$m^*A = m^*A$$

$\therefore R$ is measurable

Hence $\tilde{R} = \phi$ is also measurable.

Lemma 1:

If $m^*E = 0$ then E is measurable.

Proof

Let $A \subset R$

And Given $m^*E = 0$

we have to prove that

$$m^*A \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

Then now, $A \cap E \subseteq E$

$R \to$ Real line
$A \to$ Any set
$E \to$ subset

$$\Rightarrow m^*(A \cap E) \leq m^*E$$

$$m^*(A \cap E) \leq 0$$

$$\therefore m^*(A \cap E) = 0 \longrightarrow (1)$$

Now $A \cap \tilde{E} \subseteq A$

$$\Rightarrow m^*(A \cap \tilde{E}) \leq m^*A$$

Adding $m^*(A \cap E)$ on both sides

$$m^*(A \cap E) + m^*(A \cap \tilde{E}) \leq m^*A + m^*(A \cap E)$$

$$m^*(A \cap E) + m^*(A \cap \tilde{E}) \leq m^*A \quad [\because \text{by } (1)]$$

$$\Rightarrow m^*A \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

Hence E is measurable

Hence the proof

Lemma 2:

If E_1 and E_2 are measurable then $E_1 \cup E_2$ is also measurable.

Proof

Let A be any set in R

It is enough to prove that

$$m^*A \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap \widetilde{(E_1 \cup E_2)}]$$

Since E_1 and E_2 are measurable,

we have

$$m^*A = m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1) \longrightarrow (1)$$

$$m^*A = m^*(A \cap E_2) + m^*(A \cap \tilde{E}_2) \longrightarrow (2)$$

Replacing A by $A \cap \tilde{E}_1$ in (2)

$$m^*(A \cap \tilde{E}_1) = m^*[(A \cap \tilde{E}_1) \cap E_2] + m^*[(A \cap \tilde{E}_1) \cap \tilde{E}_2]$$

Then now

$$\begin{aligned} A \cap (E_1 \cup E_2) &= (A \cap E_1) \cup (A \cap E_2) \\ &= (A \cap E_1) \cup (A \cap E_2 \cap R) \\ &= (A \cap E_1) \cup [(A \cap E_2) \cap (\tilde{E}_1 \cup \tilde{E}_2)] \\ &= (A \cap E_1) \cup [((A \cap E_2) \cap \tilde{E}_1) \cup ((A \cap E_2) \cap \tilde{E}_2)] \\ &= (A \cap E_1) \cup [(A \cap E_2) \cap \tilde{E}_1] \cup [(A \cap E_2) \cap \tilde{E}_2] \end{aligned}$$

$$\Rightarrow m^*[A \cap (E_1 \cup E_2)] \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap \tilde{E}_1)$$

Adding $m^*(A \cap (\tilde{E}_1 \cap \tilde{E}_2))$ on both sides

$$m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (\tilde{E}_1 \cap \tilde{E}_2)) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap \tilde{E}_1) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$$

$$m^*[A\cap(E_1\cup E_2)] + m^*[A\cap(\tilde{E}_1\cap\tilde{E}_2)] \le m^*(A\cap E_1) + m^*(A\cap\tilde{E}_1)$$

$$\le m^*A \quad [\because \text{by } ①]$$

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$$\therefore m^*A \ge m^*[A\cap(E_1\cup E_2)] + m^*[A\cap(\widetilde{E_1\cup E_2})]$$

Hence $E_1\cup E_2$ is measurable

Hence the proof

Corollary 3:

A family m of measurable sets is an algebra of sets.

Proof

If $E_1, E_2 \in m$

we know that

If E_1 and E_2 are measurable then $E_1\cup E_2$ is also measurable.

$\therefore$ we get $E_1\cup E_2 \in m$

Also, If E_1 is measurable then $\tilde{E}_1$ is also measurable

(i.e) $E_1 \in m \Rightarrow \tilde{E}_1 \in m$

$\therefore$ Hence m is an algebra of sets

Hence the proof

Note

If $E_1, E_2, \ldots, E_n$ are measurable then

$E_1 \cup E_2 \cup \ldots \cup E_n$ is also measurable.

Lemma 3

Let A be any set and $E_1, E_2 \ldots E_n$ be a finite sequence of disjoint measurable set then $m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i)$.

Proof

We have to prove this Lemma by induction on "n"

When $n=1$ the result is true

Assume that the result is true for $n-1$ sets.

$$\text{(i.e) } m^*\left[A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right] = \sum_{i=1}^{n-1} m^*(A \cap E_i) \rightarrow (1)$$

Given that $E_1, E_2, \ldots E_n$ are disjoint measurable sets

consider

$$A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n = A \cap E_n \longrightarrow (2)$$

And then now

$$A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap \tilde{E}_n = A \cap [(E_1 \cup E_2 \cup \ldots E_n) \cap \tilde{E}_n]$$

$$= A \cap [(E_1 \cap \tilde{E}_n) \cup (E_2 \cap \tilde{E}_n) \cup \ldots \cup (E_n \cap \tilde{E}_n)]$$

$$= A \cap [E_1 \cup E_2 \cup \ldots \cup E_{n-1} \cup \phi]$$

$$A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap \tilde{E}_n = A \cap \left(\bigcup_{i=1}^{n-1} E_i\right) \longrightarrow (3)$$

Since E_n is measurable, where A is any subset in R

24 $$m^*A = m^*(A\cap E_n) + m^*(A\cap \tilde{E}_n) \longrightarrow (4)$$

Replace A by $A\cap(\bigcup_{i=1}^{n} E_i)$ in (4)

$$m^*[A\cap(\bigcup_{i=1}^{n} E_i)] = m^*[A\cap(\bigcup_{i=1}^{n} E_i)\cap E_n] + m^*[A\cap(\bigcup_{i=1}^{n} E_i)$$

Sub (2) and (3) we get

$$m^*[A\cap(\bigcup_{i=1}^{n} E_i)] = m^*(A\cap E_n) + m^*[A\cap(\bigcup_{i=1}^{n-1} E_i)]$$

$$= m^*(A\cap E_n) + m^*[(A\cap E_1)\cup(A\cap E_2)\cup \cdots \cup(A\cap E_{n-1})]$$

$$m^*[A\cap(\bigcup_{i=1}^{n} E_i)] = \sum_{i=1}^{n} m^*(A\cap E_i)$$

Thus the result for any "n"

Hence the proof

Corollary 4

Take $A = R$ in the above inequality lemma we get

$$m^*[R\cap(\bigcup_{i=1}^{n} E_i)] = \sum_{i=1}^{n} m^*(R\cap E_i)$$

$$m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(E_i)$$

Theorem 4

U.C The collection m of measurable sets is a σ- algebra.

25 proof

we have to prove that the complement of measurable set is measurable

And the union of countable collection of measurable set is also measurable.

Let $A_n \in m$, $n = 1, 2, 3 \ldots\ldots$

And $E = \bigcup_{i=1}^{\infty} A_i$

claim :.

There exist a sequence $E_n \in m$, $n = 1, 2, \ldots$ Such that $E_i \cap E_j = \phi$ for $i \neq j$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} E_i$$

proof of the claim,

Set $E_1 = A_1$ for $n > 1$ define

$$E_n = A_n - (A_1 \cup A_2 \cup \ldots \cup A_{n-1})$$

$$= A_n \cap (A_1 \cup A_2 \cup \ldots \cup A_{n-1})^{\sim}$$

$$E_n = A_n \cap (\tilde{A}_1 \cap \tilde{A}_2 \cap \ldots \cap \tilde{A}_{n-1}) \rightarrow ①$$

Since m is an algebra

$\tilde{A}_i \in m$ for $i = 1, 2 \ldots n-1$

and $A_n \cap \tilde{A}_1 \cap \ldots \cap \tilde{A}_{n-1} \in m$

$\Rightarrow$ $E_n \in m$ $\forall n$ and $E_n \subset A_n$

Let $m \neq n$

Suppose that $m < n$

26 we have $E_m \subset A_m$

$$\Rightarrow E_m \cap E_n \subset A_m \cap E_n \rightarrow (2)$$

Then now

$$A_m \cap E_n = A_m \cap [A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \dots \cap \tilde{A}_{n-1}]$$

$$= (A_m \cap A_n) \cap (A_m \cap \tilde{A}_1) \cap \dots \cap (A_m \cap \tilde{A}_m) \cap (A_m \cap \tilde{A}_{n-1})$$

$$A_m \cap E_n = \phi$$ since $m < n$

From (2) we have

$$E_m \cap E_n = \phi$$

Similarly, we can prove for $m > n$

we have $E_i \subset A_i$

$$\Rightarrow \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} A_i \rightarrow (3)$$

Let $x = \bigcup_{i=1}^{\infty} A_i$

$\Rightarrow x \in A_i$ for some i

Let n be the smallest value of i such that $x \in A_i$

Then $x \notin A_i$ for $i < n$

(i.e) $x \in \tilde{A}_i$ for $i < n$

$\therefore x \in \tilde{A}_{n-1} \cap \dots \cap \tilde{A}_1$

$x \in A_n \cap \tilde{A}_{n-1} \cap \cdots \cap \tilde{A}_1$

2 7

(i.e) $x \in E_n$

$$\Rightarrow \quad x \in \bigcup_{i=1}^{\infty} E_i$$

$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} E_i \rightarrow (4)$$

From (3) and (4)

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} E_i$$

Hence the claim

Thus if E is the union of countable collection of measurable set it must be the union of pairwise disjoint measurable sets.

Let $F_n = \bigcup_{i=1}^{n} E_i$

Then F_n is measurable

Also $F_n \subset E$

And $\tilde{F}_n \supset \tilde{E}_n$

$$\Rightarrow A \cap \tilde{F}_n \supset A \cap \tilde{E}_n \rightarrow (5)$$

where A is any set

we have

$$m^* A = m^*(A \cap F_n) + m^*(A \cap \tilde{F}_n)$$

$$m^* A \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) + m^*(A \cap \tilde{E}_n)$$

$$m^*A \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

Since the left side of inequality is independent of "n".

we have,

$$m^*A \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

$$\geq m^*(A \cap \bigcup_{i=1}^{\infty} E_i) + m^*(A \cap \tilde{E})$$

$$\geq m^*(A \cap \bigcup_{i=1}^{\infty} A_i) + m^*(A \cap \tilde{E}) \quad [\because \text{From claim}]$$

$$m^*A \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

$\therefore$ E is measurable

Thus $E = \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$

Hence $\mathcal{M}$ is a σ-algebra

Hence the proof

Lemma 4 :

The interval (a,∞) is measurable

proof

Let A be any subset of R

we have to prove that

$$m^*A \geq m^*[A \cap (a,\infty)] + m^*[A \cap (a,\infty)^\sim]$$

Let $A_1 = A \cap (a,\infty)$

$A_2 = A \cap (a,\infty)^\sim$

$= A \cap (-\infty, a]$

(i.e) To prove

$$m^*A \geq m^*A_1 + m^*A_2$$

If $m^*A = \infty$ there is nothing to prove

If $m^*A < \infty$

Then given $\epsilon > 0$ there exist a countable collection of open intervals of $\{I_n\}$ such that $A \subset \cup I_n$

And $\sum l(I_n) \leq m^*A + \epsilon \rightarrow (1)$

Let $I_n' = I_n \cap (a, \infty)$

$I_n'' = I_n \cap (-\infty, a)$

then I_n' and I_n'' are intervals (are may be empty)

$$l(I_n) = l(I_n') + \lambda(I_n'')$$

$$\Rightarrow m^* l(I_n) = m^* l(I_n') + m^* \lambda(I_n'') \rightarrow (2)$$

Since $A_1 \subset \cup I_n$

$$\Rightarrow A \cap (a, \infty) \subset \cup I_n \cap (a, \infty)$$

$$A_1 \subset \cup I_n'$$

$$m^*A_1 \leq \sum \lambda(I_n') \rightarrow (3)$$

Similarly, $m^*A_2 \leq \sum \lambda(I_n'') \rightarrow (4)$

Adding ③ and ④

$$m^*A_1 + m^*A_2 \leq \Sigma\lambda(I_n') + \Sigma\lambda(I_n'')$$

$$\leq \Sigma[\lambda(I_n') + \lambda(I_n'')]$$

$$\leq \Sigma\lambda(I_n)$$

$$m^*A_1 + m^*A_2 \leq m^*A + \epsilon$$

Since ϵ is arbitrary

$$\Rightarrow m^*A_1 + m^*A_2 \leq m^*A$$

(i.e) $m^*A \geq m^*A_1 + m^*A_2$

$$\Rightarrow m^*A \geq m^*[A \cap (a,\infty)] + m^*[A \cap (a,\infty)^c]$$

$\therefore (a,\infty)$ is measurable

Hence the proof

Borel set

The collection $\mathcal{B}$ of Borel sets is the smallest σ - Algebra which contains all of the open sets.

Theorem 5

Every borel set is measurable. In particular each open set and each closed set is measurable.

Proof

It is enough to show that $\mathfrak{m}$ contains all of open sets.

Since m is a σ - Algebra

And $\mathcal{B}$ is the smallest σ - algebra containing all open sets of R.

we get $\mathcal{B} \subseteq m$ and hence the proof will be as follows,

claim :

"Every open set R is measurable"

Consider (a,∞) for any $a \in R$

we have proved (a,∞) is measurable

$\therefore (a,\infty) \in m$

Since m is a σ - algebra

$(a,\infty)^c = (-\infty, a] \in m$

(i.e) $(-\infty, a]$ is measurable

For any $b \in R$

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \tfrac{1}{n}]$$

Since each of $(-\infty, b - 1/n]$ is measurable

And since countable union of measurable set is countable

we get $(-\infty, b)$ is measurable

Any open interval (a,b) can be written as, $(a,b) = (a,\infty) \cap (-\infty, b)$

Since both (a,∞) and $(-\infty,b)$ are measurable we get (a,b) is measurable.

we know that

Each open set is the union of Countable number of open intervals

Hence each open set in R is measurable,

Thus m is a σ-algebra containing the open set and must therefore contains the family $\mathcal{B}$ of Borel sets.

Hence every Borel set is measurable.

Hence the proof

Lebesgue Measure

The Lebesgue measure m is the set function from the family $\mathfrak{M}$ of Lebesgue measure set to the extended real line. defined by

$$mE = m^*E$$

(i.e) $m : \mathfrak{M} \to R$ defined by $mE = m^*E$

$\therefore$ m is the restriction of m^* to $\mathfrak{M}$

Preposition 4 :

Let $\{E_i\}$ be a sequence of measurable set then $m(\cup E_i) \leq \sum mE_i$ if the sets E_n

are pairwise disjoint measurable sets then

33 $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} mE_i$

proof

Let $E_1, E_2, \ldots E_n$ be a finite sequence of measurable sets. then

we know that (Lemma 3)

Let A be any set and $E_1, E_2, \ldots E_n$ be a finite sequence of disjoint measurable sets then

$$m^*[A \cap (\bigcup_{i=1}^{n} E_i)] = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Then we have

$$m(A \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m(A \cap E_i) \rightarrow ①$$

where A is any subset of R

Let $A = R$ then

$$m(R \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m(R \cap E_i)$$

$$m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m(E_i) \rightarrow ②$$

Thus m is finitely additive

Let $\{E_i\}$ be a infinite sequence of pairwise disjoint measurable sets

Then $\bigcup_{i=1}^{\infty} E_i \supseteq \bigcup_{i=1}^{n} E_i$

$$m(\bigcup_{i=1}^{\infty} E_i) \geq m(\bigcup_{i=1}^{n} E_i)$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{n} m(E_i) \quad [\text{by } ②]$$

Since this inequality is true for every "n", we have

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m(E_i) \rightarrow ③$$

Since m is countable sub additive

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m(E_i) \rightarrow ④$$ (see [illegible]

From ③ and ④

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

Hence the proof

Preposition 5 :

Let $\{E_n\}$ be a infinite decreasing sequence of measurable sets.

✓ (i.e) A sequence with $E_{n+1} \subset E_n$ for each n.

Let mE_1 be finite $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n$.

Proof

Given that $\{E_n\}$ is a decreasing sequence of measurable sets.

(i.e) $E_{n+1} \subset E_n \ \forall n$

Let $E = \bigcap_{i=1}^{\infty} E_i$ and Let $F_i = E_i - E_{i+1}$

$F_1 = E_1 - E_2$

$F_2 = E_2 - E_3$

Claim

$$E_1 - E = \bigcup_{i=1}^{\infty} F_i$$

(35) where F_i is a pairwise disjoint

Proof of the claim :-

Let $x \in E_1 - E$

$\Rightarrow x \in E_1$ and $x \notin E = \bigcap_{i=1}^{\infty} E_i$

(i.e) $x \in E_1$ and $x \notin E_i$ for some i

Let "j" be the smallest suffix such that $x \notin E_j$

Then $x \in E_{j-1}$, $x \notin E_j \Rightarrow x \in \tilde{E}_j$

$\Rightarrow x \in E_{j-1} \cap \tilde{E}_j$

(i.e) $x \in E_{j-1} - E_j$

Now, $x \in F_{j-1}$

$x \in \bigcup_{j=1}^{\infty} F_j$

$\Rightarrow x \in \bigcup_{i=1}^{\infty} F_i$.

$\therefore E_1 - E \subseteq \bigcup_{i=1}^{\infty} F_i \rightarrow (1)$

Let $x \in \bigcup_{i=1}^{\infty} F_i$

(i.e) $x \in F_i$ for some i

$x \in E_i - E_{i+1}$

$\Rightarrow x \in E_i$ and $x \notin E_{i+1}$

$\therefore x \notin \bigcap_{i=1}^{\infty} E_i = E$

Then $x \notin E \Rightarrow x \in \tilde{E}$

(36)

Since $E_1 \supset E_i$ and $x \in E_1$

$\therefore x \in E_1 \cap \tilde{E}$

Now $x \in E_1 - E$

$\therefore \bigcup_{i=1}^{\infty} F_i \subseteq E_1 - E \rightarrow (2)$

From (1) and (2)

$$E_1 - E = \bigcup_{i=1}^{\infty} F_i$$

To prove F_i are pairwise disjoint

without loss of generality

Let $i < j$

$\therefore E_i \supset E_j \rightarrow E_j \subset E_i$

If $x \in F_i$ then $x \in E_i - E_{i+1}$

$\Rightarrow x \in E_i$ and $x \notin E_{i+1}$

Since $E_{i+1} \supset E_j$ (contains) $\Rightarrow E_{j+1} \subset E_j \subset E_{i+1} \subset E_i$

we have $x \notin E_j$

$\Rightarrow x \notin F_j$ $\rightarrow$ $F_i = E_i - E_{i+1}$

If $x \in F_j$ then $x \in E_j - E_{j+1}$

$\Rightarrow x \in E_j$ and $x \notin E_{j+1}$

[If $x \in F_i$

$\Rightarrow x \in E_i$ and $x \notin E_{i+1}$]

Since $E_i \supset E_{i+1} \supset E_j \supset E_{j+1}$

$\Rightarrow x \notin E_j$

which is a contradiction

$x \notin F_i$

$F_i \cap F_j = \phi$

$\therefore F_i$ is a pairwise disjoint

Hence the claim

Since $F_i = E_i - E_{i+1}$

$= E_i \cap \tilde{E}_{i+1}$

we have F_i's are measurable

Also $E_1 - E = \bigcup_{i=1}^{\infty} F_i$, F_i are disjoint

$m(E_1 - E) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$

$= \sum_{i=1}^{\infty} mF_i$

$\therefore m(E_1 - E) = \sum_{i=1}^{\infty} m(E_i - E_{i+1}) \rightarrow (3)$

Since $E \subset E_1$

we have $E_1 = E \cup (E_1 - E)$ is a disjoint union.

Since $E = \cap E_i$

E is measurable and $E_1 - E$ is also measurable.

$\therefore m(E_1) = m(E) + m(E_1 - E) \rightarrow (4)$

Similarly, $E_{i+1} \subset E_i$

we get

$$mE_i = mE_{i+1} + m(E_i - E_{i+1}) \quad \forall i \rightarrow (5)$$

eqn (4) becomes

$$mE_1 - mE = m(E_1 - E)$$

$$= \sum_{i=1}^{\infty} m(E_i - E_{i+1}) \quad [\because \text{from } (3)]$$

$$= \sum_{i=1}^{\infty} (mE_i - mE_{i+1})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n-1} (mE_i - mE_{i+1})$$

$(E_1 - E_2) + (E_2 - E_3) + (E_3 - E_4) + \dots + (E_{n-1} - E_n)$

$$= \lim_{n\to\infty} (mE_1 - mE_n)$$

$$mE_1 - mE = mE_1 - \lim_{n\to\infty} mE_n$$

$$\Rightarrow \quad mE = \lim_{n\to\infty} mE_n$$

Since $E = \bigcap_{i=1}^{\infty} E_i$

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n$$

Hence the proof

Measurable Function

An extended real value function f is said to be measurable if its domain is measurable

And its satisfies one of the following four statements

For each real number α

1). $\{x : f(x) > \alpha\}$ is measurable

2). $\{x : f(x) \geq \alpha\}$ is measurable

3). $\{x : f(x) < \alpha\}$ is measurable

4). $\{x : f(x) \leq \alpha\}$ is measurable

Preposition 6 :

Let f be a extended real valued function whose domain is measurable. Then the following Statements are equivalent

For each real number α

domain is measurable then the function is measurable.

1). $\{x : f(x) > \alpha\}$ is measurable

2). $\{x : f(x) \geq \alpha\}$ is measurable

3). $\{x : f(x) < \alpha\}$ is measurable

4). $\{x : f(x) \leq \alpha\}$ is measurable

5). $\{x : f(x) = \alpha\}$ is measurable

Proof

Let the domain of f be "D"

① $\Rightarrow$ ④

we Know that

$$\{x : f(x) \leq \alpha\} = D - \{x : f(x) > \alpha\}$$

Since the difference of two measurable sets is measurable

$\Rightarrow \{x : f(x) \leq \alpha\}$ is measurable

Hence the proof

Similarly, we can prove

4 $\Rightarrow$ 1, 2 $\Rightarrow$ 3, 3 $\Rightarrow$ 2

case 2 : 1 $\Rightarrow$ 2

Next know that

$$\{x : f(x) \geq \alpha\} = \bigcap_{i=1}^{\infty} \{x : f(x) > \alpha - \tfrac{1}{n}\}$$

Each of the set $\{x : f(x) > \alpha - \frac{1}{n}\}$ is measurable by 1

Also, intersection of sequence of measurable set is measurable

$\therefore$ The set $\{x : f(x) \geq \alpha\}$ is measurable

Hence the proof

case 3 : 2 $\Rightarrow$ 1

Next know that

$$\{x : f(x) > \alpha\} = \bigcup_{i=1}^{\infty} \{x : f(x) \geq \alpha + \tfrac{1}{n}\}.$$ is measurable. by 2

Also the union of the sequence of

measurable set is also measurable

$\therefore \{x : f(x) > \alpha\}$ is measurable

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Hence the proof

~~Similarly, we can prove~~ $\cancel{③ \Rightarrow ④}$

$\therefore$ The four statements are equivalent

case (A) Assume any one and hence all the four of them hold.

Now, we have

$\{x : f(x) = \alpha\} = \{x : f(x) \leq \alpha\} \cap \{x : f(x) \geq \alpha\}$ is measurable,

Since the intersection of two measurable set is measurable

case 4 $\therefore$ ②, ④ $\Rightarrow$ ⑤

Suppose $\alpha = \infty$ then

$$\{x : f(x) = \infty\} = \bigcap_{i=1}^{\infty} \{x : f(x) \geq n\}$$

$\Rightarrow$ since the intersection

$\therefore$ ② $\Rightarrow$ ⑤

Similarly, for $\alpha = -\infty$

we have ④ $\Rightarrow$ ⑤

Hence the proof

Example

1. Constant functions are measurable

Solution

Let $f : D \to R \cup (-\infty, \infty)$

ble

we defined by $f(x) = C$

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where D is measurable and $D \subset R$

Now then

$$\{x : f(x) > \alpha\} = \begin{cases} D & \text{when } \alpha < C \\ \phi & \text{when } \alpha \geq C \end{cases}$$

Since D and ϕ are measurable → empty set is measurable

we have

$\{x : f(x) > \alpha\}$ is measurable

$\therefore f$ is measurable

Hence constant function is measurable

2) If A is the measurable subset of R then ψ_A (characteristic function) is measurable

Solution

we know that

$$\psi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then

$$\{x : f(x) > \alpha\} = \begin{cases} R & \text{when } \alpha < 0 \\ A & \text{when } 0 \leq \alpha < 1 \\ \phi & \text{when } \alpha \geq 1 \end{cases}$$

Since R, A and ϕ are measurable

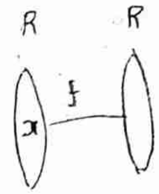
we have $\{x : f(x) > \alpha\}$ is measurable

$\therefore f$ is measurable

Hence ψ_A is measurable.

3. Continuous function from $R \to R$ are measurable

Solution 43



Suppose f is Continuous

Then $\{x : f(x) > \alpha\} = f^{-1}(\alpha, \infty)$

Since (α, ∞) is measurable and inverse image Of open set is open.

w.k.T open set is measurable so

we have $f^{-1}(\alpha, \infty)$ is measurable

$\therefore f$ is measurable

(i.e) Continuous function is measurable

Preposition : 7

Let c be a constant and f, g are two measurable functions defined on the same domain.

Then prove that the function $f+c$, cf,

Proof

1) To prove $f+c$ is measurable

It is enough to show that

$\{x : f(x) + c < \alpha\}$ is measurable

Now

$\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$

Since f is measurable

44 $\Rightarrow \{x : f(x) < \alpha - c\}$ is measurable

constant

definition of measurable $\{x : f(x) < \alpha\}$

Hence $\{x : f(x) + c < \alpha\}$ is measurable

2) To prove cf is measurable

If $c = 0$ then $cf = 0$

which is a constant function

$\therefore$ cf is measurable

If $c \neq 0$ then

$$\{x : cf(x) < \alpha\} = \begin{cases} x : f(x) < \alpha/c & \text{if } c > 0 \\ x : f(x) > \alpha/c & \text{if } c < 0 \end{cases}$$

In any case the R.H.S is measurable

$\therefore$ $\{x : cf(x) < \alpha\}$ is measurable

$\therefore$ cf is measurable.

3) To prove $g - f$ is measurable

$$g - f = g + (-1)f$$

Here $(-1)f$ is measurable

Since cf is measurable

$\therefore$ $-f$ is measurable

Hence $g - f$ is measurable

4) To prove $f + g$ is measurable

Consider

$$\{x : (f+g)x < \alpha\} = \{x : f(x) + g(x) < \alpha\}$$

$\{x : (f+g)x < \alpha\} = \{x : f(x) < \alpha - g(x)\}$

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Since between any two real number there exist a rational number say r such that

$f(x) < r < \alpha - g(x) \Rightarrow f(x) < r,\ r < \alpha - g(x)$

$f(x) < r,\ g(x) < \alpha - r$

$\therefore \{x : (f+g)x < \alpha\} = \cup \{\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}\}$

x க்கு element f(x), g(x) இரண்டையும் satisfy பண்ணக்கூடிய தான் அந்த set இரண்டிலும் இருக்கும் so $\cap$

x க்கு 1, 2 ... value கொடுக்கும் தால் union

Since rationals are countable and since each of $\{x : f(x) < \alpha\}$ and $\{x : g(x) < \alpha - r\}$ are measurable for every "r".

we get $\{x : (f+g)(x) < \alpha\}$ is measurable.

(i.e) $f+g$ is measurable.

5) To prove fg is measurable

we shall prove that f^2 is measurable.

$\{x : f^2(x) < \alpha\} = \{x : f(x) < \sqrt{\alpha}\} \cup \{x : f(x) > -\sqrt{\alpha}\}$

Each set in R.H.S is measurable.

And Hence $\{x : f^2(x) < \alpha\}$ is measurable.

$\therefore f^2$ is measurable

Now $fg = \dfrac{(f+g)^2 - (f-g)^2}{4}$

$\therefore fg$ is measurable.

Limit superior and Limit inferior

46 If $\{f_n\}$ is a sequence of function we defined its limit superior by

$$\overline{\lim} f_n = \inf_{n} \sup_{K \geq n} f_K$$

we defined its limit inferior by

$$\underline{\lim} f_n = \sup_{n} \inf_{K \geq n} f_K$$

Theorem 6

Let $\{f_n\}$ be a sequence of measurable functions then the following functions are measurable.

1). $\sup\{f_1, f_2 \dots f_n\}$

2). $\inf\{f_1, f_2 \dots f_n\}$

3) $\sup_{n} f_n$

4) $\inf_{n} f_n$

5). $\overline{\lim} f_n = \inf_{n} \sup_{K \geq n} f_K$

6) $\underline{\lim} f_n = \sup_{n} \inf_{K \geq n} f_K$

Proof

1. Let $h = \sup\{f_1, f_2 \dots f_n\}$

(i.e) $h(x) = \sup\{f_1(x), f_2(x), \dots f_n(x)\}$

we

we 47

we shall show that

$$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

Let $h(x) > \alpha$

$\Rightarrow \sup f_i(x) > \alpha$

(i.e) Then there exist an "i" such that $f_i(x) \xrightarrow{\text{any one } i > \alpha} > \alpha$.

$$\therefore x \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

Let $x \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

Then there exist an "i" such that $f_i(x) > \alpha$

since $h(x) \geq f_i(x) \quad \forall i$ $\quad h(x) = \sup\{f_1(x), f_2(x) \ldots f_n(x)\}$

$h(x) \geq f_i(x)$

$h(x) > \alpha$

$$\therefore \{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

since R.H.S is an union of measurable sets it is measurable.

$\therefore \{x : h(x) < \alpha\}$ is measurable

(i.e) $h(x)$ is measurable.

2) Let $g = \inf\{f_1, f_2 \ldots f_n\}$

(i.e) $g(x) = \inf\{f_1(x), f_2(x), \ldots f_n(x)\}$

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We shall show that

$$\{x : g(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$$

Let $h(x) < \alpha$

$\Rightarrow \inf f_i(x) < \alpha$

(i.e) Then there exist an "i" such that $f_i(x) < \alpha$

$$\therefore x \in \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$$

Let $x \in \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$

Then there exist an "i" such that $f_i(x) < \alpha$

since $g(x) \leq f_i(x) \quad \forall i$

$g(x) < \alpha$

$$\therefore \{x : g(x) < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) < \alpha\}$$

Since R.H.S is a union of measurable sets it is measurable.

$\therefore \{x : g(x) < \alpha\}$ is measurable

(i.e) $g(x)$ is measurable.

3) Let $g_1(x) = \sup_n f_n(x)$ y means $\sup\{f_1(x), f_2(x) \ldots\}$

see (i) proof last line

Then $\{x : g_1(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

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$\Rightarrow g_1$ is measurable.

4). Let $g_2(x) = \inf\limits_{n} f_n(x)$

Then $\{x : g_2(x) < \alpha\} = \bigcup\limits_{i=1}^{n} \{x : f_i(x) < \alpha\}$.

$\Rightarrow g_2$ is measurable

5) we know that

$$\overline{\lim} f_n = \inf_n \sup_{k \geq n} f_k$$

Superimum is measurable so அதுபோல infimum [illegible] measurable

$$= \inf_n \sup \{f_n, f_{n+1}, \ldots\}$$

$\therefore \overline{\lim} f_n$ is measurable.

6) we know that

$$\underline{\lim} f_n = \sup_n \inf_{k \geq n} f_k$$

$$= \sup_n \inf \{f_n, f_{n+1} \ldots\}$$

$\therefore \underline{\lim} f_n$ is measurable

Hence the proof

problems

1. Suppose f and g are two real value measurable functions on the same domain and $g(x) \neq 0$ for every x in the domain. Then prove that f/g is measurable.

able

Solution

Let $\mathcal{D}$ be the domain

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Given that $g(x) \neq 0 \quad \forall\, x \in \mathcal{D}$

Let $E_1 = \{x : g(x) > 0\}$ and

$E_2 = \{x : g(x) < 0\}$

Then E_1, E_2 are measurable. when $\alpha > 0$

consider

$$\left\{x : \frac{1}{g(x)} > \alpha\right\} = \left\{x : g(x) < 1/\alpha\right\} \cap E_1$$

Then R.H.S is measurable

$\therefore \left\{x : \frac{1}{g(x)} > \alpha\right\}$ is measurable

when $\alpha < 0$

$$\left\{x : \frac{1}{g(x)} > \alpha\right\} = \left\{x : \frac{1}{g(x)} > 0\right\} \cup \left\{x : \alpha < \frac{1}{g(x)} < 0\right\}$$

$$= E_1 \cup \left[\left\{x : \frac{1}{g(x)} > \alpha\right\} \cap E_2\right]$$

$$= E_1 \cup \left[\left\{x : g(x) < \frac{1}{\alpha}\right\} \cap E_2\right]$$

Then R.H.S is measurable

$\therefore \left\{x : \frac{1}{g(x)} > \alpha\right\}$ is measurable.

when $\alpha = 0$

$$\left\{x : \frac{1}{g(x)} > 0\right\} = \left\{x : g(x) > 0\right\}$$

$$= E_1$$

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Since E_1 is measurable

$\therefore \left\{ x : \frac{1}{g(x)} > 0 \right\}$ is measurable

$\Rightarrow \frac{1}{g(x)}$ is measurable

Hence $f/g = f(1/g)$ is measurable.

Hence the proof

Almost Every where

A property is said to hold almost every where (a.e) if the set of points where if it fails to hold is a set of measure zero.

In particular, we say that $f = g$ almost every where if f and g have the same domain and

$$m\{x : f(x) \neq g(x)\} = 0$$

definition almost everywhere

$f \neq g$ a.e $mE = 0$

fails to hold

Preposition 8

If f is a measurable function and $f = g$ almost every where then g is measurable.

Proof

Let α be a positive real number.

Let $E = \{ x : f(x) \neq g(x) \}$

Then $mE = 0 \rightarrow$ a.e definition

If $g(x) > \alpha$

$x \in E$ and $g(x) > \alpha$ (or)

$x \notin E$ and $g(x) > \alpha$

a.e definition

$f \neq g$ a.e. $mE = 0$

$f = g$ a.e. $mE \neq 0$

Then now

$$\{x : g(x) > \alpha\} = \{x \in E : g(x) > \alpha\} \cup \{x \notin E : g(x) > \alpha\}$$

$$= \{x \in E : g(x) > \alpha\} \cup \{x \notin E : f(x) > \alpha\}$$

$$= \{x \in E : g(x) > \alpha\} \cup \left[\{x : f(x) > \alpha\} - \{x \in E : f(x) > \alpha\}\right]$$

$$\{x : g(x) > \alpha\} = \{x \in E : g(x) > \alpha\} \cup \left[\{x : f(x) > \alpha\} \cap \{x \in E : g(x) \leq \alpha\}\right]$$

Since f is measurable

(i.e) $\{x : f(x) > \alpha\}$ is measurable

Since $\{x \in E : g(x) > \alpha\}$ and $\{x \in E : g(x) \leq \alpha\}$ are subsets of E

And $mE = 0$ those sets are measurable.

$\therefore \{x : g(x) > \alpha\}$ is measurable

Hence g is measurable.

Littlewood's First Principle

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Let E be a given set. Then the following 5 statements are equivalent.

i) E is measurable

ii) Given $\epsilon > 0$ there is a open set $O \supset E$ with $m^*(O \sim E) < \epsilon$

iii) Given $\epsilon > 0$ there is a closed set $F \subset E$ with $m^*(E - F) < \epsilon$

iv) There is a G in G_{δ} with $E \subset G$ $m^*(G - E) = 0$

v) There is an F in F_{σ} with $F \subset E$

$$m^*(E \sim F) = 0$$

Little wood's Third principle

statement

Let E be a measurable set of finite measure. Let $\{F_n\}$ be a sequence of measurable functions, defined on E.

Let F be a measurable real valued functions such that for each x in E we have $f_n(x) \to f(x)$.

Then given $\epsilon > 0$ and $\delta > 0$ there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that $x \notin A$ and all $n \geq N$

$|f_n(x) - f(x)| < \epsilon$

proof

Let $G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$

And $E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon ; n \geq N\}$

And $E_{N+1} = \bigcup_{n=N+1}^{\infty} G_n$

we have $E_{N+1} \subset E_N$

since $f_n(x) \to f(x)$, $\forall$ x in E

we get for each $x \in E$ there must be some E_N such that $x \notin E_N$.

Hence $\cap E_n = \phi$

[If $\cap E_N \neq \phi$ there exist an $x \in E$ such that $x \in E_N$ for every N. which is contradiction for our assumption]

Each G_n is a measurable set

$\therefore$ E_N is measurable for every N.

Since E is of finite measure and $E_N \subset E$

we have E_N's are finite measure

we know that

Let $\{E_n\}$ be a infinite decreasing

$n \geq N$

sequence of measurable sets.

55 (i.e) A sequence with $E_{n+1} \subset E_n$ for each n

Let mE be finite

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n$$

$\geq \epsilon_i \}$ $n \geq N$

Since $\{E_N\}$ is a infinite decreasing sequence of measurable set.

$$m\left(\bigcap_{i=1}^{\infty} E_n\right) = \lim_{n\to\infty} mE_n$$

$$\lim_{n\to\infty} mE_n = m(\phi) = 0$$

$$\lim_{n\to\infty} mE_n = 0$$

Hence given $\epsilon > 0$ there exist an N such that

that

$$mE_n < \delta \quad \forall \ n \geq N$$

in

(i.e) $m\left\{x \in E : |f_n(x) - f(x)| \geq \epsilon, n \geq N\right\} < \delta$

Let this E_n be denoted by A then $mA < \delta$

and $\tilde{A} = \{x \in E : |f_n(x) - f(x)| < \epsilon \ \ \forall \ n \geq N\}$

$\subset E$

Hence the proof

5b) Show that if E is measurable set then each translate $E+y$ of E is also measurable.

proof

Since E is measurable

For each $\epsilon > 0$ there exist a open set $U \supset E$ such that $m^*(U-E) < \epsilon$

Since U is an open set

Then $U+y$ is also an open set

And $E+y \subset U+y$

Then Also $(U+y)-(E+y) = (U-E)+y \rightarrow (1)$

And $m^*(U-E+y) = m^*(U-E) < \epsilon$

$\therefore$ eqn (1) becomes

$$m^*[(U+y)-(E+y)] < \epsilon$$

Hence $E+y$ is measurable

Hence the proof