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TOPOLOGY

II M.Sc. MATHEMATICS

III SEMESTER

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UNIT - II   CONTINUOUS FUNCTIONS

Continuous Functions

The Product topology

The Metric topology

RESULTS :

① Let $f : A \to B$ Let $A_0 \subset A$ and $B_0 \subset B$

Then $A_0 \subset f^{-1}[f(A_0)]$ and that equality holds if f is injective

$f[f^{-1}(B_0)] \subset B_0$ and that equality holds if f is surjective

② Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$

for $i = 0$ and $i = 1$

Then f^{-1} preserves inclusions, unions, intersections and differences of sets.

ⓐ $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$

ⓑ $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$

ⓒ $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$

ⓓ $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$

f preserves inclusions and unions only.

ⓔ $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$

ⓕ $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$

ⓖ $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$ and that equality holds if f is injective

ⓗ $f(A_0 - A_1) \supset f(A_0) - f(A_1)$ and that equality holds if f is surjective

UNIT - II

Continuous Functions

DEF: Continuity of a Function

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Note: $f^{-1}(V)$ is the set of all points x of X for which $f(x) \in V$. It is empty if V does not intersect the image set $f(X)$ of Y.

Example: Let R denote the set of real numbers in its usual topology and let R_l denote the same set in the lower limit topology. Let $f : R \to R_l$ be the identity function. $f(x) = x$ for every real number x. Then f is not a continuous function because the inverse image of the open set $[a,b)$ of R_l equals itself which is not open in R.

But the identity function $g : R_l \to R$ is continuous because $g^{-1}(a,b) = (a,b)$ which is open in R_l.

Theorem: Let X and Y be topological spaces. Let $f : X \to Y$. Then the following are equivalent.

(i) f is continuous

(ii) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$

(iii) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(iv) For each $x \in X$ and each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subset V$

Proof: We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv) $\Rightarrow$ (i)

Claim: (i) $\Rightarrow$ (ii)

Let us assume that $f: X \to Y$ is continuous.

Let A be a subset of X.

To show that $f(\bar{A}) \subset \overline{f(A)}$

It is enough to show that $x \in f(\bar{A}) \Rightarrow x \in \overline{f(A)}$

Let $x \in \bar{A}$

$\Rightarrow f(x) \in f(\bar{A}) \longrightarrow$ ①

Let V be a neighbourhood of $f(x) = y$

Since f is continuous, $f^{-1}(V)$ is open in X.

By theorem, $x \in \bar{A}$ iff every openset containing x intersects A.

$\therefore$ $x \in \bar{A} \Rightarrow$ Every open set containing x intersects A.

$\Rightarrow f^{-1}(V) \cap A \neq \phi$

Let $y \in f^{-1}(V) \cap A$

$\Rightarrow y \in f^{-1}(V)$ and $y \in A$

$\Rightarrow f(y) \in V$ and $f(y) \in f(A)$

$\Rightarrow f(y) \in V \cap f(A)$

$\Rightarrow V \cap f(A) \neq \phi$

$\Rightarrow f(x) \in \overline{f(A)} \longrightarrow$ ② (by theorem)

$\therefore$ From ① & ② we prove that $f(\bar{A}) \subset \overline{f(A)}$

Claim: (ii) $\Rightarrow$ (iii)

Assume that for every subset A of X, $f(\bar{A}) \subset \overline{f(A)}$

To prove that for every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

Let B be a closed set in Y.

To prove that $f^{-1}(B)$ is closed in X.

Let $A = f^{-1}(B)$

We have prove that A is closed in X.

ie to prove that $A = \bar{A}$ (A is closed iff $A = \bar{A}$)

Since $A = f^{-1}(B)$, by elementary set theory,

we have $f(A) = f[f^{-1}(B)] \subset B \longrightarrow (1)$

By the definition of $\bar{A}$, we have $\bar{A} \supset A$

ie $A \subset \bar{A} \longrightarrow (2)$

To prove that $\bar{A} \subset A$

Let $x \in \bar{A}$

$$\Rightarrow f(x) \in f(\bar{A})$$

$$\subset \overline{f(A)} \quad \text{by (i)}$$

$$\subset \bar{B} \quad \text{by (1)}$$

$$= B \quad \text{since } B \text{ is closed in } Y$$

$\therefore f(x) \in B$

$$\Rightarrow x \in f^{-1}(B) = A$$

$$\Rightarrow x \in A$$

$$\Rightarrow \bar{A} \subset A \longrightarrow (3)$$

$\therefore$ From (2) and (3), we proved that $\bar{A} = A$

<u>claim:</u> (iii) $\Rightarrow$ (i)

Assume that for every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

To prove that $f: X \to Y$ is continuous.

ie to prove that $f^{-1}(V)$ is open in X whenever V is open in Y.

Let V be open in Y

To prove that $f^{-1}(V)$ is open in X

Let $B = Y - V$

Then $f^{-1}(B) = f^{-1}(Y-V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$

$\therefore$ By hypothesis, B is closed in $Y \Rightarrow f^{-1}(B)$ is closed in X

$\Rightarrow X - f^{-1}(V)$ is closed in X

$\Rightarrow f^{-1}(V)$ is open in X

$\Rightarrow f$ is continuous.

<u>Claim:</u> (i) $\Rightarrow$ (iv)

Assume that $f: X \to Y$ is continuous.

To prove that for each $x \in X$ and each neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subset V$

Let $x \in X$ and V be a neighbourhood of $f(x)$

$\Rightarrow f^{-1}(V)$ is open in X containing x since f is continuous.

Then the set $U = f^{-1}(V)$ is open in X and $x \in U$

$\Rightarrow f(U) = f[f^{-1}(V)] \subset V$

$\Rightarrow f(U) \subset V$

<u>Claim:</u> (iv) $\Rightarrow$ (i)

Assume that for each $x \in X$ and each neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subset V$

To show that $f: X \longrightarrow Y$ is continuous.

ie To show that $f^{-1}(V)$ is open in X whenever V is open in Y.

Let V be open in Y

To prove that $f^{-1}(V)$ is open in X.

Let $x \in f^{-1}(V)$

$\Rightarrow f(x) \in V$

$\therefore$ By hypothesis, there exists a neighbourhood U_x of x such that $f(U_x) \subset V$

$\Rightarrow U_x \subset f^{-1}(V)$

$\Rightarrow f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ which is open in X since U_x is open

$\Rightarrow f^{-1}(V)$ is open in X

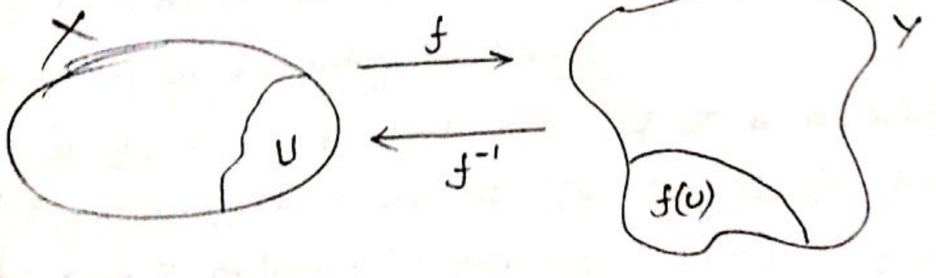
$\Rightarrow f$ is continuous.

Homeomorphism

Let X and Y be topological spaces and let $f: X \to Y$ be a bijection. If both the functions f and the inverse function $f^{-1}: Y \to X$ are continuous then f is called a homeomorphism.

Note: ① The condition that f^{-1} be continuous says that for each open set U of X, the inverse image of U under the map $f^{-1}: Y \to X$ is open in Y

② If $f: X \to Y$ is a homeomorphism then U is open in X iff $f(U)$ is open in Y.



Topological Imbedding:

Suppose that $f: X \to Y$ is an injective continuous map where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y. Then the function $f': X \to Z$ obtained by restricting the range of f is bijective. If f' is a homeomorphism of X with Z, we say that the map $f: X \to Y$ is a topological imbedding (or) simply an imbedding of X in Y.

Theorem: (Rules for constructing continuous Functions)

Let X, Y and Z be topological spaces.

(a) Constant Function:

If $f: X \to Y$ maps all of X into the single point y_0 of Y then f is continuous.

(b) Inclusion:

If A is a subspace of X, the inclusion $j: A \to X$ is continuous

(c) Composites:

If $f: X \to Y$ and $g: Y \to Z$ are continuous, then the map $g \circ f: X \to Z$ is continuous.

(d) Restricting the domain

If $f: X \to Y$ is continuous and if A is a subspace of X then the restricted function $f|A : A \to Y$ is continuous.

(e) Restricting or expanding the Range

Let $f: X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$ then the function $g: X \to Z$ obtained by restricting the range of f is continuous.

If Z is a space having Y as a subspace, then the function $h: X \to Z$ obtained by expanding the range of f is continuous.

(f) Local Formulation of continuity

The map $f: X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|U_\alpha$ is continuous for each α.

PROOF: (a) Given that $f: X \to Y$ is defined by $f(x) = y_0 \ \forall x \in X$

To prove that f is continuous.

Let V be open in Y

To prove that $f^{-1}(V)$ is open in X.

$$f^{-1}(V) = \begin{cases} X & \text{if } y_0 \in V \\ \phi & \text{if } y_0 \notin V \end{cases}$$

In each case $f^{-1}(V)$ is open in X since X and ϕ are itself open.

$\Rightarrow f$ is continuous.

(b) Given that A is a subspace of X

To prove that the inclusion $j: A \to X$ is continuous.

If U is open in X then $j^{-1}(U) = U \cap A$, which is open in A by the definition of subspace topology.

(c) Given that $f: X \to Y$ and $g: Y \to Z$ are continuous.

To prove that $g \circ f: X \to Z$ is continuous.

Let U be open in Z

To prove that $(g \circ f)^{-1}(U)$ is open in X.

Since $g: Y \to Z$ is continuous, $g^{-1}(U)$ is open in Y

Since $f: X \to Y$ is continuous, $f^{-1}(g^{-1}(U))$ is open in X

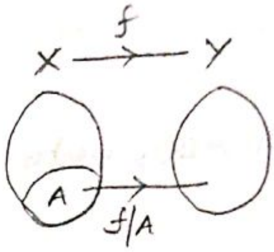
$\Rightarrow (f^{-1} \circ g^{-1})(U)$ is open in X.

$\Rightarrow (g \circ f)^{-1}(U)$ is open in X since $(g\circ f)^{-1} = f^{-1} \circ g^{-1}$

$\Rightarrow g \circ f$ is continuous.

(d) Restricting the domain

Given that $f : X \to Y$ is continuous and A is a subspace of X



To prove that $f|A : A \to Y$ is continuous.

Here $f|A = f \circ j$ where $j : A \to X$

Since f and j are continuous $\Rightarrow f \circ j$ is continuous

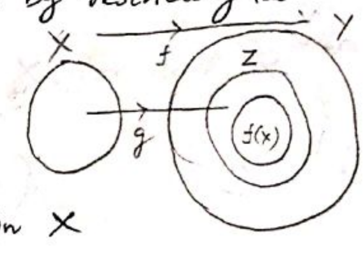
$\Rightarrow f|A$ is continuous.

(e)(i) Given that $f : X \to Y$ is continuous.

Z is a subspace of Y containing the image set $f(X)$

ie $f(X) \subset Z \subset Y$

To prove that $g : X \to Z$ obtained by restricting the range of f is continuous.



Let B be open in Z

To prove that $g^{-1}(B)$ is open in X

Since B is open in $Z \subset Y$, we have

$B = Z \cap U$ where U is open in Y

$\cancel{\Rightarrow f^{-1}(Z\cap U) = f^{-1}(B)}$

Since $f(X) \subset Z$, by elementary set theory, we have

$f^{-1}(U) = g^{-1}(B)$

But $f^{-1}(U)$ is open in X

$\Rightarrow g^{-1}(B)$ is open in X

$\Rightarrow g : X \to Z$ is continuous.

(ii) To show that $h : X \to Z$ is continuous where Y is a subspace of Z, obtained by expanding the range of f.

Here $h = j \circ f$

where $j : Y \to Z$ and $f : X \to Y$ are continuous.

$\Rightarrow j \circ f$ is continuous.

$\Rightarrow h$ is continuous.

(5) Given that $X = \bigcup_{\alpha} U_\alpha$ where U_α is open such that $f|U_\alpha$ is continuous for each α.

To prove that $f : X \to Y$ is continuous.

Let V be open in Y

To prove that $f^{-1}(V)$ is open in X.

Then $(f/U_\alpha)^{-1}(V) = f^{-1}(V) \cap U_\alpha$

Since $f|U_\alpha$ is continuous for each α

$\Rightarrow (f|U_\alpha)^{-1}(V)$ is open in X for each α

$\Rightarrow f^{-1}(V) \cap U_\alpha$ is open in X. for each α

But $f^{-1}(V) = \bigcup_{\alpha} (f^{-1}(V) \cap U_\alpha)$

$\Rightarrow f^{-1}(V)$ is open in X

$\Rightarrow f$ is continuous.

The Pasting Lemma:

Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$ then f and g combine to give a continuous function $h : X \to Y$ defined by setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

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Proof: Given $X = A \cup B$ where A and B are closed in X.

Also given that $f: A \to Y$ and $g: B \to Y$ are continuous.

If $f(x) = g(x)$ for every $x \in A \cap B$ then

we have to prove that $h: X \to Y$ is continuous.

Let C be closed in Y

To prove that $h^{-1}(C)$ is closed in X

Since $f: A \to Y$ and $g: B \to Y$ are continuous

$\Rightarrow f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B

$\Rightarrow f^{-1}(C)$ and $g^{-1}(C)$ are closed in X since A and B are closed in X.

$\Rightarrow f^{-1}(C) \cup g^{-1}(C)$ is closed in X.

Hence it is enough to prove that

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

Let $x \in h^{-1}(C) \longrightarrow (1)$

But $h^{-1}(C) \subset X = A \cup B$

$\Rightarrow x \in A \cup B$

$\Rightarrow x \in A$ (or) $x \in B$

If $x \in A$ then since $x \in h^{-1}(C)$

$\Rightarrow h(x) \in C$ where $x \in A$

$\Rightarrow f(x) \in C$ since $h(x) = f(x)$ if $x \in A$

$\Rightarrow x \in f^{-1}(C)$

If $x \in B$ then since $x \in h^{-1}(C)$

$\Rightarrow h(x) \in C$ where $x \in B$

$\Rightarrow g(x) \in C$ since $h(x) = g(x)$ if $x \in B$

$\Rightarrow x \in g^{-1}(C)$

Thus in any case $x \in f^{-1}(C) \cup g^{-1}(C) \longrightarrow (2)$

$\therefore$ From ① and ② we proved that

$$h^{-1}(c) \subset f^{-1}(c) \cup g^{-1}(c) \longrightarrow ③$$

Next to prove that

$$f^{-1}(c) \cup g^{-1}(c) \subset h^{-1}(c)$$

Let $x \in f^{-1}(c) \cup g^{-1}(c) \longrightarrow ④$

$\Rightarrow x \in f^{-1}(c)$ (or) $x \in g^{-1}(c)$

If $x \in f^{-1}(c)$ then $x \in A$ since $f^{-1}(c) \subset A$

$\therefore f(x) \in C$ where $x \in A$

$\Rightarrow h(x) \in C$ since $h(x) = f(x)$ if $x \in A$

$\Rightarrow x \in h^{-1}(c)$

If $x \in g^{-1}(c)$ then $x \in B$ since $g^{-1}(c) \subset B$

$\therefore g(x) \in C$ where $x \in B$

$\Rightarrow h(x) \in C$ since $h(x) = g(x)$ if $x \in B$

$\Rightarrow x \in h^{-1}(c)$

Thus in any case, $x \in h^{-1}(c) \longrightarrow ⑤$

$\therefore$ From ④ and ⑤, we have

$$f^{-1}(c) \cup g^{-1}(c) \subset h^{-1}(c) \longrightarrow ⑥$$

$\therefore$ From ③ and ⑥, we prove that

$$h^{-1}(c) = f^{-1}(c) \cup g^{-1}(c)$$

$f^{-1}(c)$ and $g^{-1}(c)$ are closed in A and B

$\Rightarrow f^{-1}(c) \cup g^{-1}(c)$ is closed in $A \cup B = X$

$\Rightarrow h^{-1}(c)$ is closed in X

$\Rightarrow h$ is continuous.

Topological Property :

Let X and Y be two topological spaces and P be a property of X that is entirely expressed in terms of the topology of X (that is, in terms of the open sets of X) and there is a homeomorphism $f : X \to Y$. If we can prove that the property P hold in Y then the property P is said to be a topological property.

Theorem : Maps into products

Let $f_1 : A \to X$ and $f_2 : A \to Y$ and let $f : A \to X \times Y$ be given by the equations $f(a) = (f_1(a), f_2(a))$ Then f is continuous iff f_1 and f_2 are continuous.

PROOF: Given that $f_1 : A \to X$ and $f_2 : A \to Y$ and

$f : A \to X \times Y$ is defined by $f(a) = (f_1(a), f_2(a))$

Part I Assume that f is continous.

To prove that $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections onto the first and second factors respectively.

Let U and V be open sets in X and Y

$$\pi_1^{-1}(U) = U \times Y \quad \text{and} \quad \pi_2^{-1}(V) = X \times V$$

They are open in $X \times Y$

$\Rightarrow \pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$ are open in $X \times Y$

$\Rightarrow \pi_1$ and π_2 are continuous

For each $a \in A$, $\quad f_1(a) = \pi_1(f(a)) = \pi_1 \circ f$

$\quad f_2(a) = \pi_2(f(a)) = \pi_2 \circ f$

Since π_1 and f are continuous

$\Rightarrow$ the composition map $\pi_1 \circ f$ is also continuous

$\Rightarrow f_1$ is continuous.

Since π_2 and f are continuous

$\Rightarrow$ the composition map $\pi_2 \circ f$ is also continuous.

$\Rightarrow f_2$ is continuous

$\Rightarrow f_1$ and f_2 are continuous.

Part II Assume that $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous

To prove that $f : A \to X \times Y$ is continuous.

Let $U \times V$ be a basis element in $X \times Y$ where U is open in X and V is open in Y.

To prove that $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$

Let $a \in f^{-1}(U \times V) \Leftrightarrow f(a) \in U \times V$

$\Leftrightarrow (f_1(a), f_2(a)) \in U \times V$

$\Leftrightarrow f_1(a) \in U$ and $f_2(a) \in V$

$\Leftrightarrow a \in f_1^{-1}(U)$ and $a \in f_2^{-1}(V)$

$\Leftrightarrow a \in f_1^{-1}(U) \cap f_2^{-1}(V)$

$\therefore f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$

Here U is open in X and V is open in Y.

Since $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous

$\Rightarrow f_1^{-1}(U)$ is open in A and $f_2^{-1}(V)$ is open in A

$\Rightarrow f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in A

$\Rightarrow f^{-1}(U \times V)$ is open in A

$\Rightarrow f$ is continuous.

DEF: Let J be an index set. We define a J-tuple of elements of X to be a function $x : J \to X$. If α is an element of J, we often denote the value of x at α by x_α rather than $x(\alpha)$; we call it the αth coordinate of x. and we often denote the function x itself by the symbol

$$(x_\alpha)_{\alpha \in J}$$

We denote the set of all J-tuples of elements of X by X^J

DEF: Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets. Let $X = \bigcup_{\alpha \in J} A_\alpha$. The cartesian product of this indexed family, denoted by $\prod_{\alpha \in J} A_\alpha$ is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \longrightarrow \bigcup_{\alpha \in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$

We denote the product simply by $\prod A_\alpha$ and its general element by (x_α), if the index set is understood.

If all the sets A_α are equal to one set X, then the Cartesian product $\prod_{\alpha \in J} A_\alpha$ is just the set X^J of all J-tuples of elements of X.

Box Topology:

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces.

and $\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha \;\middle|\; U_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \in J \right\}$

is a basis for a topology on product space $\prod_{\alpha \in J} X_\alpha$.

The topology generated by this basis is called the Box Topology.

PROJECTION MAPPING:

Let $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ be the function assigning to each element of the product space its βth coordinate

ie $\pi_\beta\left((x_\alpha)_{\alpha \in J} \right) = x_\beta$

It is called the projection mapping associated with the index β.

PRODUCT TOPOLOGY:

Let $\mathcal{S}_\beta$ denote the collection

$\mathcal{S}_\beta = \left\{ \pi_\beta^{-1}(U_\beta) \;\middle|\; U_\beta \text{ is open in } X_\beta \right\}$ and let

$\mathcal{S}$ denote the union of these collections

ie. $\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$

The topology generated by the subbasis $\mathcal{S}$ is called the Product Topology.

In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space

Comparision between Box and Product Topologies

The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Theorem:

Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation

$f(a) = \left(f_\alpha(a)\right)_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ for each α.

Let $\prod X_\alpha$ have the product topology. Then the function f is continuous iff each function f_α is continuous.

PROOF: Given that $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by

$$f(a) = \left(f_\alpha(a)\right)_{\alpha \in J}$$

where $f_\alpha : A \to X_\alpha$ for each α.

$\prod X_\alpha$ have the product topology

Part I Assume that f is continuous.

To prove that each function f_α is continuous.

Let $\pi_\beta : \prod X_\alpha \to X_\beta$ be the projection mapping associated with β.

Let U_β be open in X_β

To prove that $\pi_\beta^{-1}(U_\beta)$ is open in $\prod X_\alpha$

Here the set $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on $\prod X_\alpha$

$\Rightarrow \pi_\beta^{-1}(U_\beta)$ is open in $\prod X_\alpha$

$\Rightarrow \pi_\beta$ is continuous.

Now $f_\beta : \pi_\beta \circ f$

$\Rightarrow f_\beta$ is the composition of two continuous functions π_β and f

$\Rightarrow f_\beta$ is continuous.

Part II Assume that each f_α is continuous.

To prove that $f: A \to \prod_{\alpha \in J} X_\alpha$ is continuous.

Consider the basis element of product topology on $\prod X_\alpha$

Let it be $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β for each β

To prove that $f^{-1}(\pi_\beta^{-1}(U_\beta))$ is open in A.

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = (f^{-1} \circ \pi_\beta^{-1})(U_\beta)$$
$$= (\pi_\beta \circ f)^{-1}(U_\beta)$$
$$= f_\beta^{-1}(U_\beta)$$

Here U_β is open in X_β

Since $f_\beta : A \to X_\beta$ is continuous

$\Rightarrow f_\beta^{-1}(U_\beta)$ is open in A.

$\Rightarrow (\pi_\beta \circ f)^{-1}(U_\beta)$ is open in A

$\Rightarrow (f^{-1} \circ \pi_\beta^{-1})(U_\beta)$ is open in A

$\Rightarrow f^{-1} \circ (\pi_\beta^{-1}(U_\beta))$ is open in A

$\Rightarrow f$ is continuous.

Prove that the product of continuous functions need not be continuous.

PROOF: Consider R^ω, the box topology $R^\omega = \prod_{n \in Z_+} X_n$ where $X_n = R$ for each n

Define $f: R \to R^\omega$ by $f(t) = (t, t, t, \ldots\ldots)$

Here $f_n : R \to R$ by $f_n(t) = t$ is continuous.

To prove that $f: R \to R^{\omega}$ is not continuous

Let $B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots\cdots$

be the basis element which is open in R^{ω}

To prove that $f^{-1}(B)$ is not open in R

Suppose $f^{-1}(B)$ is open in R

Since $0 \in f^{-1}(B)$, $f^{-1}(B)$ is open

$\Rightarrow$ There exists an interval $(-\delta, \delta)$ such that

$$0 \in (-\delta, \delta) \subset f^{-1}(B)$$

$$\Rightarrow f(-\delta, \delta) \subset B$$

$$\Rightarrow \pi_n(f(-\delta, \delta)) \subset \pi_n(B)$$

$$\Rightarrow f_n(-\delta, \delta) \subset \pi_n(B)$$

$$\Rightarrow (-\delta, \delta) \subset (-\tfrac{1}{n}, \tfrac{1}{n}) \quad \text{since } f_n(t) = t$$

which is a contradiction

$\therefore f^{-1}(B)$ is not open in R

$\therefore$ Product of continuous functions need not be continuous.

<u>Theorem:</u> Let $\{X_\alpha\}$ be an indexed family of spaces and let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology then $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$

<u>PROOF:</u> Given that $\{X_\alpha\}$ ~~be~~ is an indexed family of spaces.

and $A_\alpha \subset X_\alpha$ for each α

$\prod X_\alpha$ is given either the product or the box topology.

To prove that $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$

Let $x = (x_\alpha) \in \prod \bar{A}_\alpha$

where $x_\alpha \in \bar{A}_\alpha$ for each α.

Let $U = \prod U_\alpha$ be a basis element for either the product or box topology that contains x.

Then since $x_\alpha \in \bar{A}_\alpha$, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$ for each α.

Then $y = (y_\alpha)$ is an element of both U and $\prod A_\alpha$

$$\Rightarrow U \cap \prod A_\alpha \neq \phi$$

$\therefore$ Every element of x intersects $\prod A_\alpha$

$$\Rightarrow x \in \overline{\prod A_\alpha}$$

Hence $\prod \bar{A}_\alpha \subset \overline{\prod A_\alpha} \longrightarrow (1)$

Conversely let us assume that $x = (x_\alpha) \in \overline{\prod A_\alpha}$

To prove that $x_\beta \in \bar{A}_\beta$ for any given index β

Let V_β be an arbitrary open set of X_β containing x_β.

$$\pi_\beta^{-1}(V_\beta) = X_{\alpha_1} \times X_{\alpha_2} \times X_{\alpha_3} \times \cdots\cdots \times V_\beta \times \cdots\cdots$$

Now $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ containg (x_α)

Since $(x_\alpha) \in \overline{\prod A_\alpha}$, $\pi_\beta^{-1}(V_\beta) \cap \prod A_\alpha \neq \phi$

Choose a point $y = (y_\alpha) \in \pi_\beta^{-1}(V_\beta) \cap \prod A_\alpha$

$\Rightarrow (y_\alpha) \in \pi_\beta^{-1}(V_\beta)$ and $(y_\alpha) \in \prod A_\alpha$

$\Rightarrow y_\beta \in V_\beta$ and $y_\beta \in A_\beta$

$\Rightarrow V_\beta \cap A_\beta \neq \phi$

$\therefore x_\beta \in \bar{A}_\beta$

$(x_\alpha) \in \prod \bar{A}_\alpha$

$\Rightarrow \overline{\prod A_\alpha} \subset \prod \bar{A}_\alpha \longrightarrow (2)$

$\therefore$ From (1) & (2), we prove that $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$

The Metric Topology:

DEF: METRIC

A Metric on a set X is a function $d: X \times X \to \mathbb{R}$ having the following properties:

(i) $d(x, y) \geq 0$ for all $x, y \in X$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

(Triangle inequality)

Note: Given a metric d on X, the number $d(x, y)$ is called the distance between x and y in the metric d.

DEF: OPEN BALL (or) ε-ball

Let (X, d) be a metric space and $x \in X$. Then the open ball centered at x and radius ε denoted by $B_d(x, \varepsilon)$ is defined as $B_d(x, \varepsilon) = \{ y \in X \mid d(x, y) < \varepsilon \}$ which is called the ε-ball centered at x.

METRIC TOPOLOGY

Let (X, d) be a metric space and let

$\mathcal{B} = \{ B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0 \}$ is a basis for the topology on X and the topology generated by this basis is known as metric topology induced by d.

Note: A set U is open in the metric topology induced by d iff for every $y \in U$ there exists a $\delta > 0$ such that $B_d(y, \delta) \subset U$

Discrete Topology:

Let X be a non-empty set and define $d: X \times X \to R$ by $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Then d is a metric on X. This metric is called discrete Metric. The topology it induces is the discrete topology, the basis element $B(x,1)$, for example, consists of the point x alone.

Standard Topology:

Let $X = R$. Define $d: R \times R \to R$ by $d(x,y) = |x-y|$. Then d is a metric on R. This metric is called usual metric.

Let $\mathcal{B} = \{ B_d(x, \varepsilon) \mid x \in R, \varepsilon > 0 \}$ is a basis for a topology on R.

The topology generated by this basis is called the Standard topology on R.

METRIZABLE:

If X is a topological space then we say that X is said to be metrizable if there exists a metric d on the set X such that d induces the topology of X. A metric space is a metrizable space X together with a specific metric d that gives the topology of X.

BOUNDED:

Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number M such that $d(a_1, a_2) \leq M$ for every pair a_1, a_2 of points of A.

DIAMETER: If A is bounded and non-empty, then the diameter of A is defined to be the number $\operatorname{diam} A = \sup \{ d(a_1, a_2) \mid a_1, a_2 \in A \}$

Theorem:

Let X be a metric space with metric d. Define $\bar{d} : X \times X \to R$ by the equation $\bar{d}(x,y) = \min\{d(x,y), 1\}$. Then $\bar{d}$ is a metric that induces the same topology as d. The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof: Given that X is a metric space with metric d.

$\bar{d} : X \times Y \to R$ defined by $\bar{d}(x,y) = \min\{d(x,y), 1\}$

To prove : $\bar{d}$ is a metric on X

(i) $\bar{d}(x,y) = \min\{d(x,y), 1\} \geq 0$ since $d(x,y) \geq 0$

(ii) TP $\bar{d}(x,y) = 0$ iff $x = y$

$$\bar{d}(x,y) = 0 \Leftrightarrow \min\{d(x,y), 1\} = 0$$
$$\Leftrightarrow d(x,y) = 0$$
$$\Leftrightarrow x = y$$

(iii) $\bar{d}(x,y) = \min\{d(x,y), 1\}$

$= \min\{d(y,x), 1\}$ since d is a metric $d(x,y) = d(y,x)$

$= \bar{d}(y,x)$

(iv) To prove : $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$

Case (i) Suppose $d(x,y) \geq 1$ (or) $d(y,z) \geq 1$

$$\bar{d}(x,y) = \min\{d(x,y), 1\} = 1$$
$$\bar{d}(y,z) = \min\{d(y,z), 1\} = 1$$
$$\therefore \bar{d}(x,y) + \bar{d}(y,z) \geq 1$$
$$\bar{d}(x,z) = \min\{d(x,z), 1\} \leq 1 \leq \bar{d}(x,y) + \bar{d}(y,z)$$
$$\therefore \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$$

Case (ii) Suppose $d(x,y) < 1$ and $d(y,z) < 1$

$$\Rightarrow \bar{d}(x,y) = \min\{d(x,y), 1\} = d(x,y)$$

and $\bar{d}(y,z) = \min\{d(y,z), 1\} = d(y,z)$

$\therefore \; \bar{d}(x,y) = d(x,y)$ and $\bar{d}(y,z) = d(y,z)$

We know that $d(x,z) \leq d(x,y) + d(y,z)$

$$= \bar{d}(x,y) + \bar{d}(y,z)$$

$$\therefore \; d(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z) \rightarrow (1)$$

But $\bar{d}(x,z) = \min\{d(x,z), 1\}$

$$\leq d(x,z)$$

$$\leq \bar{d}(x,y) + \bar{d}(y,z) \quad \text{by } (1)$$

$$\therefore \; \bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$$

$\Rightarrow \bar{d}$ is a metric on X

Hence the proof.

Note: In any metric space, the collection of ε-balls with $\varepsilon < 1$ forms a basis for the metric topology because every basis element containing x contains such an ε-ball centered at x. with $\varepsilon < 1$.

Also the collection of ε-balls with $\varepsilon < 1$ under these two metrics d and $\bar{d}$ are the same collection.

$\therefore$ Both d and $\bar{d}$ induce the same topology on X.

Lemma:

Let d and d' be two metrics on the set X. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for each $x \in X$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$

Proof: Given that d and d' are the two metrics on the set X.

$\mathcal{T}$ and $\mathcal{T}'$ be the topologies of d and d' respectively.

To prove: $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for each $x \in X$ and each $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$

Part I: Assume that $\mathcal{T}$ and $\mathcal{T}'$ be two topologies of X

Assume that $\mathcal{T}'$ is finer than $\mathcal{T}$

To prove: For each $x \in X$ and each $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$

$\therefore$ By lemma, for every $x \in X$ and $B \in \mathcal{B}$ $\exists$ a $B' \in \mathcal{B}'$ such that $x \in B' \subset B$

$\Rightarrow$ For every $x \in X$ and each basis element $B_d(x, \varepsilon)$ there exists a basis element B' for $\mathcal{T}'$ such that

$x \in B' \subset B_d(x, \varepsilon) \rightarrow$ (1)

$\therefore$ We can find a ball $B_{d'}(x, \delta) \subset B'$

$\therefore$ (1) $\Rightarrow$ $B_{d'}(x, \delta) \subset B' \subset B_d(x, \varepsilon)$

$\Rightarrow$ $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$

Part II: Assume that for each $x \in X$ and each $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon) \rightarrow$ (2)

To prove that $\mathcal{T}'$ is finer than $\mathcal{T}$

ie to prove that $\mathcal{T}' \supset \mathcal{T}$ (or) $\mathcal{T} \subset \mathcal{T}'$

Let B be the basis element of $\mathcal{T}$ containing x

$\therefore$ We can find a ball $B_d(x, \varepsilon) \subset B$

(2) $\Rightarrow$ There exists a $\delta > 0$ such that $x \in B_{d'}(x, \delta) \subset B$

$\therefore$ $\mathcal{T}'$ is finer than $\mathcal{T}$.

Euclidean Metric:

Given $x = (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^n$, we define the norm of x by the equation $\|x\| = (x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2)^{1/2}$ and we define the euclidean metric d on $\mathbb{R}^n$ by the equation

$$d(x, y) = \|x - y\| = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2\right]^{1/2}$$

Square Metric :

Let $x = (x_1, x_2, x_3, \ldots\ldots x_n)$ and $y = (y_1, y_2, y_3, \ldots\ldots, y_n)$ be the elements of R^n. Then the metric $\rho(x, y)$ is defined as follows.

$$\rho(x,y) = \text{Max}\{|x_1 - y_1|, |x_2 - y_2|, \ldots\ldots |x_n - y_n|\}$$

This metric $\rho(x, y)$ is called square metric.

Note: On the real line $R = R^1$, these two metrics coincide with the Standard metric for R

In the plane R^2, the basis element under d can be pictured as circular regions while the basis element under ρ can be pictured as square regions.

Ex: 1 In R^n, define $d'(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|$

Show that d' is a metric on R^n

Sol: To prove that d' is a metric in R^n

Let $x, y \in R^n$

$\Rightarrow x = (x_1, x_2, x_3, \ldots x_n)$ and $y = (y_1, y_2, y_3, \ldots, y_n)$

where $x_i, y_i \in R$

(i) $d'(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|$

≥ 0 since $x_i, y_i \in R \Rightarrow |x_i - y_i| \geq 0$

$$\begin{aligned} d'(x,y) = 0 &\Leftrightarrow |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| = 0 \\ &\Leftrightarrow \sum_{i=1}^{n} |x_i - y_i| = 0 \\ &\Leftrightarrow |x_i - y_i| = 0 \text{ for each } i \\ &\Leftrightarrow x_i - y_i = 0 \text{ for each } i \\ &\Leftrightarrow x_i = y_i \text{ for each } i \\ &\Leftrightarrow x = y \end{aligned}$$

(ii) $d'(x,y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|$

$= |y_1 - x_1| + |y_2 - x_2| + \cdots + |y_n - x_n|$

$= d'(y,x)$

$\therefore d'(x,y) = d'(y,x)$

(iii) To prove: $d'(x,z) \leq d'(x,y) + d'(y,z) \quad \forall x,y,z \in R^n$

Let $x, y, z \in R^n$

Take $n=2$. $\therefore x, y, z \in R^2$

$\Rightarrow x = (x_1, x_2),\ y = (y_1, y_2),\ z = (z_1, z_2)$

where $x_i, y_i, z_i \in R$ for $i = 1, 2$

$d'(x,z) = |x_1 - z_1| + |x_2 - z_2|$

$= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2|$

$\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$

$= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2|$

$= d'(x,y) + d'(y,z)$

$\therefore d'(x,z) \leq d'(x,y) + d'(y,z)$

$\therefore d'$ is a metric on R^n

Ex: 2 If $\rho(x,y) = \text{Max}\{|x_1 - y_1|, |x_2 - y_2|, \cdots |x_n - y_n|\}$ then show that ρ is a metric on R^n

Sol: (i) $\rho(x,y) = \text{Max}\{|x_i - y_i|\} \geq 0$

(ii) $\rho(x,y) = 0 \Leftrightarrow \text{Max}\{|x_i - y_i|\} = 0$

$\Leftrightarrow |x_i - y_i| = 0$ for each i

$\Leftrightarrow x_i - y_i = 0$ for each i

$\Leftrightarrow x_i = y_i$ for each i

$\Leftrightarrow x = y$

(ii) $\rho(x,y) = \text{Max}\{|x_i - y_i|\} = \text{Max}\{|y_i - x_i|\} = \rho(y,x)$

(iii) To prove : $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

$$\rho(x,z) = \text{Max}\{|x_i - z_i|\}$$
$$= \text{Max}\{|x_i - y_i + y_i - z_i|\}$$
$$\leq \text{Max}\{|x_i - y_i|\} + \text{Max}\{|y_i - z_i|\}$$
$$= \rho(x,y) + \rho(y,z)$$

$\therefore \rho(x,z) \leq \rho(x,y) + \rho(y,z)$

$\therefore \rho$ is a metric on $\mathbb{R}^n$

Theorem: The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof: Let $\mathcal{T}$, $\mathcal{T}_d$, $\mathcal{T}_\rho$ be the product topology, topology induced by d and the topology induced by ρ respectively.

To prove : $\mathcal{T}_d = \mathcal{T}_\rho = \mathcal{T}$

Claim: $\rho(x,y) \leq d(x,y) \leq \sqrt{n}\,\rho(x,y) \quad \forall\, x,y \in \mathbb{R}^n$

Let $x, y \in \mathbb{R}^n$

$\Rightarrow x = (x_1, x_2, x_3, \dots, x_n)$ and $y = (y_1, y_2, y_3, \dots, y_n)$

Consider $\rho(x,y) = \text{Max}\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}$

$$= |x_{i_0} - y_{i_0}| \quad \text{(say)}$$

Clearly $|x_{i_0} - y_{i_0}|^2 \leq |x_1 - y_1|^2 + |x_2 - y_2|^2 + \dots + |x_n - y_n|^2$

Taking squareroot on bothsides, we get

$$|x_{i_0} - y_{i_0}| \leq \{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \dots + |x_n - y_n|^2\}^{1/2}$$
$$= d(x,y)$$

$\therefore \rho(x,y) \leq d(x,y) \longrightarrow (1)$

Also $|x_1 - y_1| \leq |x_{i_0} - y_{i_0}|$, $|x_2 - y_2| \leq |x_{i_0} - y_{i_0}|$, ...

$|x_n - y_n| \leq |x_{i_0} - y_{i_0}|$

$\Rightarrow |x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2 \leq n\,|x_{i_0} - y_{i_0}|^2$

$\Rightarrow \left\{ |x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2 \right\}^{1/2} \leq n^{1/2}\,|x_{i_0} - y_{i_0}|$

$\Rightarrow d(x,y) \leq \sqrt{n}\,\rho(x,y) \longrightarrow (2)$

$\therefore$ From (1) and (2), we have

$$\rho(x,y) \leq d(x,y) \leq \sqrt{n}\,\rho(x,y)$$

Hence the claim.

Next we prove $\mathcal{T}_d = \mathcal{T}_\rho$

ie we have to show that $\mathcal{T}_d \subset \mathcal{T}_\rho$ and $\mathcal{T}_\rho \subset \mathcal{T}_d$

First to prove: $\mathcal{T}_d \subset \mathcal{T}_\rho$

ie To prove that for every $x \in \mathbb{R}^n$ and each $\varepsilon > 0$, $\exists$ a $\delta > 0$

such that $B_d(x, \delta) \subset B_\rho(x, \varepsilon)$

Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$

Choose $\delta = \varepsilon$

Then to prove that $B_d(x,\delta) \subset B_\rho(x,\varepsilon)$

Let $y \in B_d(x,\delta)$

$\Rightarrow d(x,y) < \delta = \varepsilon$

$\Rightarrow \rho(x,y) < \varepsilon$ since $\rho(x,y) \leq d(x,y)$

$\Rightarrow y \in B_\rho(x,\varepsilon)$

$\therefore B_d(x,\delta) \subset B_\rho(x,\varepsilon)$

$\Rightarrow \mathcal{T}_d \subset \mathcal{T}_\rho \longrightarrow (3)$

Next we prove that $\mathcal{T}_\rho \subset \mathcal{T}_d$

ie we have to prove that $\forall\, x \in \mathbb{R}^n$ and each $\varepsilon > 0$ $\exists$ a $\delta > 0$

such that $B_\rho(x,\delta) \subset B_d(x,\varepsilon)$

Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$

Choose $\delta = \frac{\varepsilon}{\sqrt{n}}$

To prove that $B_\rho(x, \delta) \subset B_d(x, \varepsilon)$

Let $y \in B_\rho(x, \delta)$

$\Rightarrow \rho(x, y) < \delta = \frac{\varepsilon}{\sqrt{n}}$

$\Rightarrow \sqrt{n}\, \rho(x, y) < \varepsilon$

$\Rightarrow d(x, y) < \varepsilon$

$\Rightarrow y \in B_d(x, \varepsilon)$

$\therefore B_\rho(x, \delta) \subset B_d(x, \varepsilon)$

$\Rightarrow \mathcal{T}_\rho \subset \mathcal{T}_d \longrightarrow (4)$

$\therefore$ From (3) and (4), we proved that $\mathcal{T}_\rho = \mathcal{T}_d \longrightarrow (5)$

Then to prove that $\mathcal{T} = \mathcal{T}_\rho$

i.e. we have to show that $\mathcal{T} \subset \mathcal{T}_\rho$ and $\mathcal{T}_\rho \subset \mathcal{T}$

First let us prove that $\mathcal{T}_\rho \subset \mathcal{T}$

Let $B_\rho(x, \varepsilon)$ be a basis element for $\mathcal{T}_\rho$

Let $y \in B_\rho(x, \varepsilon)$

We need to find a basis element B for $\mathcal{T}$ such that

$y \in B \subset B_\rho(x, \varepsilon)$

$B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \cdots (x_n - \varepsilon, x_n + \varepsilon)$

is a basis element for $\mathcal{T}$

$\therefore \mathcal{T}_\rho \subset \mathcal{T}$

To prove: $\mathcal{T} \subset \mathcal{T}_\rho$

Let $B = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ be a basis element for $\mathcal{T}$

Let $x = (x_1, x_2, x_3, \ldots, x_n) \in B$

Then $x_i = (a_i, b_i)$ for $i = 1, 2, 3, \ldots, n$

For each i, we can arrange an $\varepsilon_i > 0$ such that

$$(x_i - \varepsilon_i \,,\, x_i + \varepsilon_i) \subset (a_i, b_i)$$

Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n\}$

Then $B_\rho(x, \varepsilon)$ be a basis element of $\mathcal{T}_\rho$ and

$$x \in B_\rho(x, \varepsilon)$$

<u>Claim:</u> $B_\rho(x, \varepsilon) \subset B$

Let $y \in B_\rho(x, \varepsilon)$

$\Rightarrow \rho(x, y) < \varepsilon$

$\Rightarrow \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots |x_n - y_n|\} < \varepsilon$

$\Rightarrow |x_i - y_i| < \varepsilon$ for every $i = 1, 2, 3, \ldots, n$

$\Rightarrow y_i \in (x_i - \varepsilon, x_i + \varepsilon)$

$\Rightarrow y_i \in (a_i, b_i)$

$\Rightarrow y \in B$

$\therefore B_\rho(x, \varepsilon) \subset B$

$\Rightarrow \mathcal{T}_\rho \supset \mathcal{T}$

ie $\mathcal{T} \subset \mathcal{T}_\rho$

$\therefore \mathcal{T} = \mathcal{T}_\rho \longrightarrow (6)$

$\therefore$ From (5) and (6), we have $\mathcal{T}_\rho = \mathcal{T}_d = \mathcal{T}$

Let $B = (a_1, b_1) \times (a_2, b_2) \times \dots\dots \times (a_n, b_n)$ be a basis element for the product topology.

Let $x = (x_1, x_2, x_3, \dots\dots x_n)$ be an element of B

For each i, there is an ε_i such that

$$(x_i - \varepsilon_i, \ x_i + \varepsilon_i) \subset (a_i, b_i)$$

Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n\}$

To prove that $B_\rho(x, \varepsilon) \subset B$

Let $y \in B_\rho(x, \varepsilon)$

$$\Rightarrow \rho(x, y) < \varepsilon$$

$$\Rightarrow \max\{|x_1 - y_1|, |x_2 - y_2|, \dots\dots |x_n - y_n|\} < \varepsilon$$

$$\Rightarrow |x_i - y_i| < \varepsilon \text{ for all } i$$

$$\Rightarrow |y_i - x_i| < \varepsilon \ \forall i$$

$$\Rightarrow -\varepsilon < y_i - x_i < \varepsilon$$

$$\Rightarrow x_i - \varepsilon < y_i < x_i + \varepsilon \ \forall i$$

$$\Rightarrow y_i \in (x_i - \varepsilon, x_i + \varepsilon) \subset (a_i, b_i) = B$$

$$\Rightarrow y_i \in B$$

$\therefore y \in B$

$\therefore B_\rho(x, \varepsilon) \subset B$

$\therefore$ The topology induced by ρ is finer than the product topology

i.e. $\mathcal{T}_\rho \supset \mathcal{T} \longrightarrow (6)$

Conversely let $B_\rho(x, \varepsilon)$ be a basis element for each ρ-topology.

Given the element $y \in B_\rho(x, \varepsilon)$, we need to find a basis element B for the product topology such that

$y \in B \subset B_\rho(x, \varepsilon)$

To p.t. $B_\rho(x,\varepsilon) = (x_1-\varepsilon, x_1+\varepsilon) \times (x_2-\varepsilon, x_2+\varepsilon) \times \cdots \times (x_n-\varepsilon, x_n+\varepsilon)$

Let $y = (y_1, y_2, y_3, \dots y_n) \in B_\rho(x, \varepsilon)$

$y \in B_\rho(x,\varepsilon)$

$\Leftrightarrow \rho(x,y) < \varepsilon$

$\Rightarrow \max\{|x_1-y_1|, |x_2-y_2|, \dots, |x_n-y_n|\} < \varepsilon$

$\Rightarrow |x_i - y_i| < \varepsilon$ for each i

$\Rightarrow |x_1-y_1| < \varepsilon, |x_2-y_2| < \varepsilon, \dots, |x_n-y_n| < \varepsilon$

$\therefore$ As before $y_1 \in (x_1-\varepsilon, x_1+\varepsilon)$

$y_2 \in (x_2-\varepsilon, x_2+\varepsilon)$

$\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots$

$y_n \in (x_n-\varepsilon, x_n+\varepsilon)$

$\therefore B_\rho(x,\varepsilon) = (x_1-\varepsilon, x_1+\varepsilon) \times (x_2-\varepsilon, x_2+\varepsilon) \times \cdots \times (x_n-\varepsilon, x_n+\varepsilon)$

The RHS is a basis element for the product topology.

Thus the product topology is finer than the ρ-topology

ie $\mathcal{T} \supset \mathcal{T}_\rho \rightarrow (7)$

$\therefore$ From (6) and (7), we prove that $\mathcal{T} = \mathcal{T}_\rho \rightarrow (8)$

$\therefore$ From (5) & (8), we get $\mathcal{T} = \mathcal{T}_\rho = \mathcal{T}_d$

Hence the theorem

UNIFORM TOPOLOGY:

Let J be an index set and $a, b \in R$. Define $\bar{d}(a,b)$ by $\bar{d}(a,b) = \min\{1, |a-b|\}$. Then $\bar{d}$ is a metric on R.

This type of metric is called standard bounded metric on R.

$$R^J = R \times R \times R \times \cdots\cdots = \{(x_\alpha)_{\alpha \in J}\}$$

Let $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ be the elements of R^J.

Define $\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$

Then $\bar{\rho}$ is a metric on R^J

This type of metric is called uniform metric on R^J.

The topology induced by uniform metric $\bar{\rho}$ on R^J is called the uniform topology on R^J

Theorem:

The uniform topology on R^J is finer than the product topology and coarser than the box topology.

PROOF: To prove that the uniform topology on R^J is finer than the product topology on R^J

Let $\prod U_\alpha$ be a basis element for the product topology containing the point $x = (x_\alpha)_{\alpha \in J}$

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the indices for which $U_\alpha \neq R$

Since U_{α_i} is open in R, we can choose $\varepsilon_i > 0$ such that

$$B_{\bar{d}}(x_i, \varepsilon_i) \subset U_{\alpha_i} \longrightarrow (1)$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_n\}$

Let $B_{\bar{\rho}}(x, \varepsilon)$ be a basis element for the uniform topology.

Claim: $B_{\bar{\rho}}(x, \varepsilon) \subset \prod U_\alpha$

Let $z \in B_{\bar{\rho}}(x, \varepsilon)$

$\Rightarrow \bar{\rho}(x, z) < \varepsilon$

$\Rightarrow \sup\{\bar{d}(x_\alpha, z_\alpha) \mid \alpha \in J\} < \varepsilon$

$\Rightarrow \bar{d}(x_\alpha, z_\alpha) < \varepsilon$

$\Rightarrow \bar{d}(x_{\alpha_i}, z_{\alpha_i}) < \varepsilon_i$ since $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n\}$

$\Rightarrow z_{\alpha_i} \in B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$ by 1

$\Rightarrow z_{\alpha_i} \in U_{\alpha_i}$

$\Rightarrow z = (z_\alpha)_{\alpha \in J} \in \prod U_\alpha$

$\Rightarrow z \in \prod U_\alpha$

$\Rightarrow$ Uniform topology on $\mathbb{R}^J$ is finer than the product topology on $\mathbb{R}^J$

Next to prove that the uniform topology on $\mathbb{R}^J$ is coarser than the box topology.

Let $B_{\bar{\rho}}(x, \varepsilon)$ be a basis element for the uniform topology.

Let $U = \prod (x_\alpha - \frac{1}{2}\varepsilon, x_\alpha + \frac{1}{2}\varepsilon)$ be a basis element for the box topology.

Claim: $B_{\bar{\rho}}(x, \varepsilon) \supset U$

Let $y \in U$

$\Rightarrow y_\alpha \in (x_\alpha - \frac{1}{2}\varepsilon, x_\alpha + \frac{1}{2}\varepsilon)$

$\Rightarrow |x_\alpha - y_\alpha| < \frac{1}{2}\varepsilon$

$\Rightarrow \min\{|x_\alpha - y_\alpha|, 1\} < \frac{1}{2}\varepsilon$

$\Rightarrow \bar{d}(x_\alpha, y_\alpha) < \frac{1}{2}\varepsilon$ for all α

$\Rightarrow \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\} < \frac{1}{2}\varepsilon$

$\Rightarrow \bar{\rho}(x,y) < \frac{1}{2}\varepsilon < \varepsilon$

$\Rightarrow y \in B_{\bar{\rho}}(x,\varepsilon)$

$\therefore B_{\bar{\rho}}(x,\varepsilon) \supset U$

Hence the uniform topology on R^J is coarser than the box topology.

Theorem: Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on R. If x and y are two points of R^ω, define $D(x,y) = \operatorname{Sup}\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$. Then D is a metric that induces the product topology on R^ω

(OR)

R^ω is metrizable with respect to the product topology.

PROOF: Given that $x, y \in R^\omega$

Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$

$$D(x,y) = \operatorname{Sup}\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$$

To prove that D is a metric on R.

Let $x = (x_i)_{i\in Z_+}$, $y = (y_i)_{i\in Z_+}$, $z = (z_i)_{i\in Z_+}$

(i) $D(x,y) = \operatorname{Sup}\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\} \geq 0$ since $\bar{d}$ is a metric

(ii) $D(x,y) = 0 \Leftrightarrow \operatorname{Sup}\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\} = 0$

$\Leftrightarrow \frac{\bar{d}(x_i,y_i)}{i} = 0$

$\Leftrightarrow \bar{d}(x_i,y_i) = 0$

$\Leftrightarrow x_i = y_i$

$\Leftrightarrow x = y$

(iii) $D(x,y) = \text{Sup}\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\} = \text{Sup}\left\{\frac{\bar{d}(y_i,x_i)}{i}\right\} = D(y,x)$

(iv) To prove that $D(x,z) \leq D(x,y) + D(y,z)$

Since $\bar{d}$ is a metric

$\Rightarrow \bar{d}(x_i,z_i) \leq \bar{d}(x_i,y_i) + \bar{d}(y_i,z_i)$

$$\Rightarrow \frac{\bar{d}(x_i,z_i)}{i} \leq \frac{\bar{d}(x_i,y_i)}{i} + \frac{\bar{d}(y_i,z_i)}{i}$$

$$\Rightarrow \text{Sup}\left\{\frac{\bar{d}(x_i,z_i)}{i}\right\} \leq \text{Sup}\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\} + \text{Sup}\left\{\frac{\bar{d}(y_i,z_i)}{i}\right\}$$

$\Rightarrow D(x,z) \leq D(x,y) + D(y,z)$

$\therefore$ D is a metric on R^ω

<u>To prove:</u> D induces the product topology.

ie To prove that Metric Topology is equal to product topology.

(i) <u>To prove:</u> Product Topology is finer than metric topology.

Let U be a basis element for the metric topology

and $x \in U$

We can choose an ε-ball, $B_D(x,\varepsilon) \subset U$

Choose N large enough such that $\frac{1}{N} < \varepsilon$

Finally, let V be a basis element for the product topology

given by $V = (x_1-\varepsilon, x_1+\varepsilon) \times \cdots\cdots \times (x_N-\varepsilon, x_n+\varepsilon) \times R \times R \times \cdots\cdots$

<u>Claim:</u> $V \subset U$

Let $y = (y_i)_{i\in Z_+} \in V$

$\Rightarrow y_i \in (x_i-\varepsilon, x_i+\varepsilon)$ for $i = 1,2,3,\ldots,N$ and $y_i \in R$ for $i > N$

$\Rightarrow |x_i - y_i| < \varepsilon$ for $i = 1,2,3,\ldots,N$

$\Rightarrow \min\{|x_i-y_i|, 1\} < \varepsilon$

$\Rightarrow \bar{d}(x_i, y_i) < \varepsilon$ for $i = 1, 2, 3, \dots, N$

$\Rightarrow \dfrac{\bar{d}(x_i, y_i)}{i} < \dfrac{\varepsilon}{i} < \varepsilon$ for $i = 1, 2, 3, \dots, N$

$\Rightarrow \dfrac{\bar{d}(x_i, y_i)}{i} < \varepsilon$ for $i = 1, 2, 3, \dots, N \longrightarrow (1)$

Let $i \geq N$

Then $\frac{1}{i} \leq \frac{1}{N} < \varepsilon$

Now $\bar{d}(x_i, y_i) = \min\{|x_i - y_i|, 1\} \leq 1$

$\Rightarrow \dfrac{\bar{d}(x_i, y_i)}{i} \leq \dfrac{1}{i} \leq \dfrac{1}{N}$

ie $\dfrac{\bar{d}(x_i, y_i)}{i} \leq \dfrac{1}{N}$ for $i \geq N$

$\therefore D(x, y) \leq \max\left\{\dfrac{\bar{d}(x_1, y_1)}{1}, \dfrac{\bar{d}(x_2, y_2)}{2}, \dots, \dfrac{\bar{d}(x_N, y_N)}{N}, \dfrac{1}{N}\right\}$

$\therefore$ By eqn (1), we have $D(x, y) < \varepsilon$

$\therefore y \in B_D(x, \varepsilon)$

$\Rightarrow y \in U$ since $B_D(x, \varepsilon) \subset U$

$\therefore V \subset U$

$\Rightarrow$ The product topology is finer than the metric topology.

Next to prove that the metric topology is finer than the product topology.

Let $U = \prod_{i \in Z_+} U_i$ be a basis element for the product topology and $x \in U$

Then U_i is open in R for $i = \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ and $U_i = R$ for all other indices i

Given $x \in U$, we find an open set V of the metric topology such that $x \in V \subset U$

Choose $\varepsilon_i \leq 1$ such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i$ for $i = \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$

Let $\varepsilon = \min\left\{ \frac{\varepsilon_i}{i} \mid i = \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \right\}$

Now $B_D(x, \varepsilon)$ is a basis element for the metric topology and $x \in B_D(x, \varepsilon)$

<u>Claim:</u> $B_D(x,\varepsilon) \subset U$

Let $y = (y_i)_{i \in Z_+} \in B_D(x, \varepsilon)$

$\Rightarrow D(x,y) < \varepsilon$

$\Rightarrow \frac{\bar{d}(x_i, y_i)}{i} < \varepsilon$ for every $i = \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ and $\varepsilon < \frac{\varepsilon_i}{i}$

$\therefore \frac{\bar{d}(x_i, y_i)}{i} < \frac{\varepsilon_i}{i}$ for $i = \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$

$\Rightarrow \bar{d}(x_i, y_i) < \varepsilon_i$ for $i = \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$

$\Rightarrow |x_i - y_i| < \varepsilon_i$ for $i = \alpha_1, \alpha_2, \dots, \alpha_n$

Since $\bar{d}(x_i, y_i) = \min\{|x_i - y_i|, 1\}$

$\Rightarrow y_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i$ for $i = \alpha_1, \alpha_2, \dots, \alpha_n$

$\Rightarrow y \in \prod U_i = U$

$\Rightarrow y \in U$

$\therefore B_D(x, \varepsilon) \subset U$

Hence the metric topology is finer than the product topology.

$\therefore D$ is a metric which induces the product topology on R^ω

Theorem: Let $f: X \to Y$ and let X and Y be metrizable with metrics d_X and d_Y respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

PROOF: Given that $f: X \to Y$ where X and Y are metrizable with metrics d_X and d_Y respectively.

Part I: Assume that f is continuous.

To prove that given $x \in X$ and given $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$

Let $x \in X$ and $\varepsilon > 0$ be given

Since $x \in X \Rightarrow f(x) \in Y$

Consider an open ball with centre $f(x)$ and radius ε in Y.

Then $f^{-1}(B_{d_Y}(f(x), \varepsilon))$ is open in X and contains x

There exists some δ-ball such that $B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \varepsilon))$

Let $y \in B_{d_X}(x, \delta)$

$\Rightarrow d_X(x, y) < \delta$

Also $y \in f^{-1}(B_{d_Y}(f(x), \varepsilon))$

$\Rightarrow d_Y(f(x), f(y)) < \varepsilon$

$\therefore d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$

Part II Conversely, let us assume that $x \in X$ and given $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$

To prove that $f: X \to Y$ is continuous.

Let V be open in Y containing $f(x)$ ie $f(x) \in V$

$\Rightarrow \exists$ a basis element $B_{d_Y}(f(x), \varepsilon)$ such that

$f(x) \in B_{d_Y}(f(x), \varepsilon) \subset V$

$\Rightarrow f^{-1}(B_{d_Y}(f(x), \varepsilon)) \subset f^{-1}(V) \longrightarrow (1)$

ie given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $B_{d_X}(x, \delta)$ is open in X.

If $y \in B_{d_X}(x, \delta)$ then $d_X(x, y) < \delta$

$\Rightarrow d_Y(f(x), f(y)) < \varepsilon$ by assumption

$\Rightarrow f(y) \in B_{d_Y}(f(x), \varepsilon)$

$\therefore f(y) \in B_{d_Y}(f(x), \varepsilon)$ $\forall$ $y \in B_{d_X}(x, \delta)$

$\Rightarrow f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \varepsilon)$

$\Rightarrow B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \varepsilon))$

$\Rightarrow B_{d_X}(x, \delta) \subset f^{-1}(V)$ by (1)

ie $x \in B_{d_X}(x, \delta) \subset f^{-1}(V)$ where $B_{d_X}(x, \delta)$ is a basis element. from the metrizable topological space.

$\therefore f^{-1}(V)$ is open in X.

$\Longrightarrow f$ is continuous.

DEF: A sequence $\{x_1, x_2, x_3, \dots, x_n\}$ of points of X is said to converge to the point x of X if for every neighbourhood U of x there exists a positive integer N such that x_i lies in U for all $i \geq N$

Theorem: (Sequence lemma)

Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is metrizable.

PROOF: Let X be a topological space and $A \subset X$

Given that $\{x_n\}$ is a sequence of points of A which converge to x.

Claim: $x \in \bar{A}$

Since $\{x_n\}$ converges to x where $x_n \in A$

ie x is a limit point of the sequence $\{x_n\}$

$\Rightarrow$ Every neighbourhood of x intersects A and contains infinitely many points of A

$\Rightarrow$ $x \in \bar{A}$ by theorem

Conversely suppose that X is metrizable with metric d.

Let $x \in \bar{A}$

To prove that $\{x_n\}$ converges to x

Since $x \in \bar{A} \Rightarrow$ every neighbourhood of x intersects A

Let $B_d(x, \frac{1}{n})$ be a neighbourhood of x for each positive integer n.

$\therefore B_d(x, \frac{1}{n}) \cap A \neq \phi \quad \forall n$

Let $x_n \in B_d(x, \frac{1}{n}) \cap A \quad \forall n$

$\Rightarrow x_n \in A$ and $x_n \in B_d(x, \frac{1}{n})$

Let U be open in X containing x

$\Rightarrow$ There exists a basis element $B_d(x, \varepsilon)$ such that

$x \in B_d(x, \varepsilon) \subset U$

Choose N such that $\frac{1}{N} < \varepsilon$

For $i \geq N$, we have $x_N \in A \cap B_d(x, \frac{1}{N})$

$\Rightarrow x_N \in B_d(x, \frac{1}{N})$

$\Rightarrow d(x_N, x) < \frac{1}{N} < \varepsilon$

ie $x_N \in B_d(x, \frac{1}{N}) \subset U \quad \forall i \geq N$

$\Rightarrow x_i \in U \quad \forall i \geq N$

$\therefore \{x_n\}$ converges to x

Theorem: Let $f: X \to Y$ and let X be metrizable. Then the function f is continuous iff for every convergent sequence $\{x_n\} \to x$ in X, the sequence $f(x_n) \to f(x)$ in Y.

PROOF: Let $f: X \to Y$ and X be metrizable.

Assume that f is continuous.

To prove: $\{x_n\}$ converges to $x \Rightarrow f(x_n)$ converges to $f(x)$ in Y.

Let $\{x_n\}$ converges to x in X

Claim: $\{f(x_n)\}$ converges to $f(x)$ in Y

Let $x \in X$. Then $f(x) \in Y$

Let V be a neighbourhood of $f(x)$ in Y.

$\because f(x) \in V \Rightarrow x \in f^{-1}(V)$ where $f^{-1}(V)$ is open in X.

Since $\{x_n\} \to x \Rightarrow x_i \in f^{-1}(V) \quad \forall\, i \geq N$

$\Rightarrow f(x_i) \in V \quad \forall\, i \geq N$

$\therefore \{f(x_n)\} \to f(x)$ in Y

Conversely, let $f: X \to Y$ be any function where X is metrizable.

Assume that $\{x_n\} \to x$ in $X \Rightarrow \{f(x_n)\} \to f(x)$ in Y.

To prove: f is continuous.

Let $x \in \bar{A}$

ie to prove that if $A \subset X$ then $f(\bar{A}) \subset \overline{f(A)}$ by theorem

$\Rightarrow$ There is a sequence $\{x_n\}$ of points of A converges to x

$\Rightarrow \{x_n\} \in A \longrightarrow x$ by sequence lemma

$\Rightarrow \{f(x_n)\} \longrightarrow f(x)$ by hypothesis

Since $f(x_n) \in f(A) \Rightarrow f(x) \in \overline{f(A)}$ by sequence lemma

Hence $f(\bar{A}) \subset \overline{f(A)}$

DEF: Converges uniformly

Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence $\{f_n\}$ converges uniformly to the function $f : X \to Y$ if given $\varepsilon > 0$, there exists an integer N such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and all x in X.

Theorem: (UNIFORM LIMIT THEOREM)

Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $\{f_n\}$ converges uniformly to f then f is continuous.

PROOF: Given that $f_n : X \to Y$ is a sequence of continous functions. Assume $\{f_n\}$ converges uniformly to f

To prove: f is continuous.

Let V be an open set in Y.

To prove: $f^{-1}(V)$ is open in X

Let x_0 be a point of $f^{-1}(V)$

We wish to find a neighbourhood U of x_0 such that $f(U) \subset V$.

Since V is open and $f(x_0) \in V$, we can choose an ε-ball $B(f(x_0), \varepsilon) \subset V$.

Since $\{f_n\}$ converges to f uniformly

$\Rightarrow$ given $\varepsilon > 0$ there exists an integer N such that

$$d(f_n(x), f(x)) < \frac{\varepsilon}{3} \to (1) \quad \forall\ n \geq N \text{ and all } x \in X$$

Since f_N is continuous $\Rightarrow$ we can choose a neighbourhood U of x_0 such that $f(U) \subset V$

$$\Rightarrow f_N(U) \subset B\left(f_N(x_0), \frac{\varepsilon}{3}\right) \longrightarrow (2)$$

Claim: $f(U) \subset B(f(x_0), \varepsilon)$

Let $x \in U$

Then $d(f(x), f_N(x)) < \frac{\varepsilon}{3}$ by ①

$d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}$ by ②

$d(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3}$ by ①

Now
$$
\begin{aligned}
d(f(x), f(x_0)) &\le d(f(x), f_N(x)) \\
&\quad + d(f_N(x), f_N(x_0)) \\
&\quad + d(f_N(x_0), f(x_0)) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}
$$

$\therefore d(f(x), f(x_0)) < \varepsilon$

$\Rightarrow f(x) \in B(f(x_0), \varepsilon)$

$\Rightarrow f(U) \subset B(f(x_0), \varepsilon)$

Hence the claim

$\therefore f(U) \subset V$

$\therefore f$ is continuous.

---

① Prove that R^ω in the box topology is not metrizable.

Sol: We shall show that the sequence lemma does not hold for R^ω.

Let A be a subset of R^ω consisting of those points all of whose coordinates are positive.

$A = \{(x_1, x_2, x_3, \dots) \mid x_i > 0 \text{ for all } i \in Z_+\}$

Let 0 be the origin in R^ω

ie The point $(0,0,0,\cdots,)$ each of whose coordinates is zero.

In the box topology, $0 \in \bar{A}$

If $B = (a_1, b_1) \times (a_2, b_2) \times \cdots\cdots$ is any basis element containing 0, then B intersects A

For instance, the point $(\frac{1}{2}b_1, \frac{1}{2}b_2, \cdots) \in B \cap A$

We prove that there is no sequence of points of A converging to 0.

For let (a_n) be a sequence of points of A where $a_n = \{x_{1n}, x_{2n}, x_{3n}, \cdots, x_{in}, \cdots\}$

Each coordinate x_{in} is positive.

So we can construct a basis element B' for the box topology on R by setting

$$B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \cdots\cdots$$

Then B' contains the origin 0, but it contains no member of the sequence $\{a_n\}$.

The point $a_n \notin B'$ because its nth coordinate x_{nn} does not belong to the interval $(-x_{nn}, x_{nn})$.

Hence the sequence $\{a_n\}$ cannot converge to 0 in the box topology.

Thus R^ω in the box topology is not metrizable.